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# THE ELEMENTS

# OF

# PLANE TRIGONOMETRY.

BY

EUGENE L. RICHARDS,

ASSISTANT PROFESSOR OF MATHEMATICS IN YALE COLLEGE.

NEW YORK:
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# PREFACE.

In this short work there are presented such parts of Plane Trigonometry as are required for the course of Mathematical study in the Undergraduate Academical Department of Yale College.

In preparing the work, it has been the main purpose of the writer to make the subject plain to beginners, and especially plain to that class of students to whom Mathematics are distasteful. For this purpose, a great deal of space has been given to the consideration of the elementary definitions. For this purpose, also, the definitions have been applied to acute angles and to obtuse angles separately, and, after the student is thus made familiar with them, have been extended to angles of any magnitude. It has been hoped that, in this way, even the non-mathematical mind might be easily prepared for a clear understanding of this important and interesting subject.

It is not necessary that all the chapters should be taught in the order in which they have been written. For instance, Chapter VIII., with the exception of some of the examples, might be given to the student immediately after Chapter II.

In the solution of numerical examples by logarithms, six-place tables have been used.

The references are to Todhunter's "Euclid" and to Chauvenet's "Geometry."

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THE

# ELEMENTS OF PLANE TRIGONOMETRY.

## CHAPTER I.

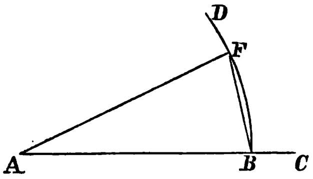
### SOME INSTRUMENTS USED IN DRAWING.

For the right representation of the magnitudes of which Trigonometry treats, the following instruments will be found useful: 1. A *pair* of *dividers* or *compasses*, having one leg to which can be adjusted a pencil, or, if necessary, an ink-point; 2. A *plane-scale*.

One of the simplest scales, and one which is sufficient for all ordinary drawing purposes, is six inches in length, and is made of ivory or of box-wood. It contains a line or lines of chords, and also lines of equal parts, of which the units are of different lengths.

(a) A *line of chords* is used for setting off angles. It consists of a line, marked from 10° to 90°, and contains the chords of all arcs from 1° up to 90°. As the chord of 60° is equal to radius (Euc. 16, IV. Cor. Ch. 5, V.), when we wish to lay down an angle of a given number of degrees we describe a circle, from a given point, with a radius equal to the length of the line measured from the beginning of the line of chords to the division marked 60; then taking the chord of the number of degrees whose angle is required as a radius, we place it in the circle. If the extremities of the chord be joined to the centre of the circle, the chord will subtend the angle required.

Thus, suppose at a given point in a given line we wish to make an angle of 25°.



Let $A$ be the given point in the given line $A\,C$, at which it is required to make an angle of 25°.

From the line of chords we take a chord of 60° as radius, and with it, from $A$ as a centre, describe an arc $B\,D$. Then from $B$ as a centre, with the line $B\,F$, as a radius, equal to a chord of 25°, taken from the *same* line of chords, we describe an arc intersecting the former arc at $F$. Join $B\,F$ and $A\,F$. Then $F\,A\,B$ is an angle of 25°, since it is at the centre of the circle, and subtends an arc of 25°.

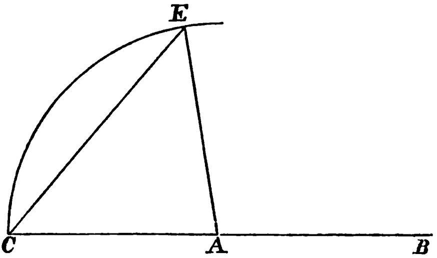
If it is required, at a given point, to make an obtuse angle of a given number of degrees, we produce the line through the given point, and make an angle on the line produced equal to the supplement of the given angle.

Thus let it be required to make an angle of 100° at the point $A$ in the given line $A\,B$.

Produce the line $A\,B$ through $A$. Make $A\,C$ equal to a chord of 60° taken from the line of chords. From

*A* as centre, with *A C* as radius, describe an arc *C E*. From *C* as centre, with a radius *C E*, equal to a chord of 80°, taken from the *same* line of chords, describe an arc cutting the former arc at *E*. Then *C A E* is an angle of 80°. But as the angles *C A E* and *E A B* are together equal to two right angles, or 180°, *E A B* is equal to 100°.

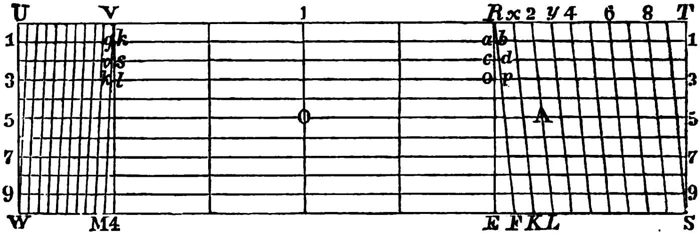
Also an obtuse angle of a certain number of degrees can be made by making two adjacent acute angles, the sum of which is equal to the given obtuse angle.



(Angles of a certain number of degrees and *minutes*, or degrees, *minutes*, and *seconds*, can be made only approximately correct by the line of chords.)

(b) A *line of equal parts* is used for drawing lines of a given length, one of the equal parts being taken as the unit, or as ten units, or as a hundred units, if the line is measured on the decimal system. Lines of equal parts are also given, by means of which feet and inches can be represented. The unit is then divided into twelve equal parts.

To obtain hundredths of the unit on the plane-scale a device is adopted which will be understood by the following figure:



In the above figure, which represents a part of a plane-scale in which the unit of measurement is an inch, let $RS$ be a square of which each side is an inch. Let $RT$ be divided into ten equal parts, $Rx$, $x2$, etc.; also let $ES$ and $ST$ be divided into ten equal parts. Through the points of division of $ST$ let lines be drawn parallel to $RT$ or $ES$. Let a straight line be drawn from $R$ to the first point ($F$) of division in the line $ES$. Let $xk$ be drawn from the first point of $RT$ to the second point of $ES$, another line be drawn from the second point of $RT$ to the third point of $ES$, and so on.

Since $Rx$, $x2$, $2y$, etc., are equal and parallel to $FK$, $KL$, etc., $RF$, $xK$, $2L$, etc., are parallel (Euc. 33, I. Ch. 32, I.). Therefore, the distances on the parallels to $RT$, through 1 and 3, etc., intercepted between any two consecutive lines, is equal to the distances on the same parallels between any other consecutive two (Euc. 34, I. Ch. 30, I.), and each of these distances is equal to $\frac{1}{10}$ of an inch.

Again, in the triangle $REF$, since $ab$ is parallel to

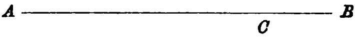
$EF$, we have, from similar triangles, $\frac{ab}{EF} = \frac{Ra}{RE} = \frac{1}{10}$; therefore $ab = \frac{EF}{10}$; but $EF = \frac{1}{10}$ of an inch; therefore $ab = \frac{1}{10}$ of $\frac{1}{10}$ of an inch, or $\frac{1}{100}$ of an inch.

In a similar manner it can be proved that $CD$ equals $\frac{2}{100}$ of an inch, $OP$ equals $\frac{3}{100}$ of an inch, etc.

If, on the left of the figure, we take $UV$ equal to $\frac{1}{2}$ inch and divide it into ten equal parts, and divide $W4$ into ten equal parts, and draw a line from $V$ to the first point $M$ of $W4$, another line from the first point of $UV$ to the second point of $W4$, and so on, we can prove $gh$ equal to $\frac{1}{100}$ of $\frac{1}{2}$ inch or $\frac{1}{200}$ of an inch, and $rs$ equal to $\frac{2}{100}$ of $\frac{1}{2}$ inch or $\frac{2}{200}$ of an inch, etc.

Suppose, now, we are required to represent a straight line of 125 feet, on the scale of 100 feet to the inch.

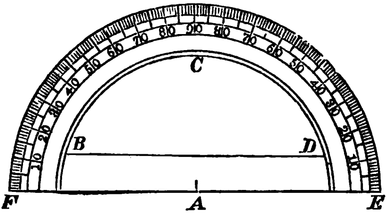
We draw any line $AB$; then placing one point of the dividers at $O$, on the plane-scale, and bringing the other to the point $A$, the intersection of the two lines 5 5 and $2L$, we lay off on $AB$ a line $AC$ equal to $OA$. Then $AC$ is the required line, as it is equal to 1 inch and $\frac{25}{100}$ inch, or to $\frac{125}{100}$ of an inch.



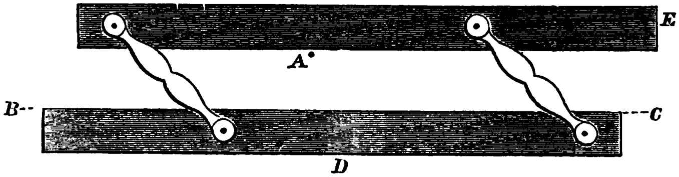
The two instruments described above are all that are *necessary* for drawing lines of a given length, or laying down angles of a given number of degrees. The following, however, are very convenient instruments:

A *protractor*, for laying down angles, and a *parallel* ruler for drawing parallel lines.

A *protractor* is a semicircle, usually of brass or some other metal, a part of the semicircle being cut out, as the part *B C D*. The outer rim left is divided into degrees, and sometimes into half degrees or even smaller parts, the lines of division all converging to a notch (*A*) on the diameter of the semicircle. The degrees are numbered from both left to right and right to left, from 10° to 170°, or from 10° to 90°, as in the figure.



At a given point in a given line, to lay down an angle of a given number of degrees by means of the protractor, place the notch, *A*, at the given point, and the line *F E* coincident with the given line. Then put a dot, or point, opposite the given number of degrees, on the outer or inner edge of the protractor. A line, drawn from this point to the given point, will make with the given straight line the required angle.



A *parallel rule* consists of two rules joined together by two pieces of metal of equal length, at equal distances apart.

To draw a line through a given point parallel to a given line, place one edge of one of the rulers coincident with the given line, and one edge of the other ruler on the point, and draw the line through the given point. It will be parallel to the given line.

Thus, suppose $A$ to be the given point and $B\ C$ the given straight line. Make the edge of the ruler, $D$, coincide with $B\ C$, and place the edge of the rule, $E$, on the point $A$. A line drawn through $A$ will be parallel to $B\ C$.

The parallel line might also have been drawn by placing the upper edge of $E$ on the point, or by placing the lower edge of $D$ coincident with the given line $B\ C$.

**Example 1. Draw the right-angled triangle of which the hypotenuse is 236 feet, and one acute angle is 30°, and find by measurement the base and perpendicular.** ***Ans.*** **Perpendicular = 118; base = 204, nearly.**

**2. Draw the triangle whose two sides are, respectively, 120 and 80, and the included angle 42°, and find by measurement the third side.**

***Ans.*** **81, almost.**

**3. Draw the right-angled triangle whose sides about the right angle are each 345, and find by measurement the hypotenuse.**

**4. Draw the triangle of which one side is 421, and the adjacent angles are 35° and 72°, and find by measurement the other sides.**

# CHAPTER II.

## PLANE TRIGONOMETRY.—DEFINITIONS.

Article 1. Trigonometry is that branch of mathematics which treats of *angles, the relations of different angles to each other, and the relations of angles to lines, surfaces, and solids.*

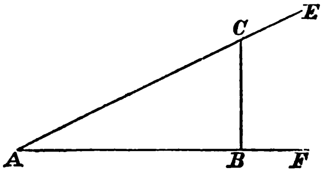
2. In Plane Trigonometry the *angles* considered are such as are contained by *straight lines*, and the *surfaces* considered are *plane* surfaces.

3. The Trigonometrical Ratios of an *angle are ratios between the sides of a right-angled triangle, in which, or with respect to which, the angle has a certain position.*

The *right-angled triangle* is called *the triangle of reference.* It is formed by *the lines containing the angle* and a *perpendicular* to one of the lines, or to one of the lines produced, drawn from any point in the other line.

The trigonometrical ratios are six in number.

4. We will first consider the *trigonometrical ratios* of an *acute angle.*



Let $A$ be any acute angle, formed by the two lines $AE$ and $AF$, meeting at the point $A$.

In $AE$ take any point

$C$, and from $C$ draw $CB$ at right angles to $AF$. Then there is formed the right-angled triangle $ABC$, the *triangle of reference.* In this triangle, calling $CB$, the *side opposite* the given angle, the *perpendicular*, and $AB$, the *side adjacent* to the given angle, the *base*—

(1.) The SINE of an angle is the ratio of the *perpendicular* to the *hypotenuse.*

Thus, Sine of $A$ (written sin. $A$) $= \dfrac{BC}{AC}$.

(2.) The TANGENT of an angle is the ratio of the *perpendicular* to the *base.*

Thus, Tangent of $A$ (written tan. $A$) $= \dfrac{BC}{AB}$.

(3.) The SECANT of an angle is the ratio of the *hypotenuse* to the *base.*

Thus, Secant of $A$ (written sec. $A$) $= \dfrac{AC}{AB}$.

(4.) The COSINE of an angle is the ratio of the *base* to the *hypotenuse.*

Thus, Cosine of $A$ (written cos. $A$) $= \dfrac{AB}{AC}$.

(5.) The COTANGENT of an angle is the ratio of the *base* to the *perpendicular.*

Thus, Cotangent of $A$ (written cot. $A$, or cotan. $A$) $= \dfrac{AB}{BC}$.

(6.) The COSECANT of an angle is the ratio of the *hypotenuse* to the *perpendicular.*

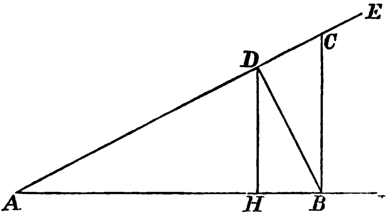
Thus, Cosecant of $A$ (written cosec. $A$) $= \dfrac{AC}{BC}$.

5. If we take $C$ as the acute angle, $AB$ as the per-

pendicular, and $BC$ as the base, we shall have sin. $C = \frac{AB}{AC}$, tan. $C = \frac{AB}{BC}$, and sec. $C = \frac{AC}{BC}$. Now, $C$ is the complement of $A$, as their sum equals 90°. Comparing the cos. $A$ with the sin. $C$, it will be seen that they are identical, both being $\frac{AB}{AC}$. Also, tan. $C$ is identical with cotan. $A$, and sec. $C$ with cosec. $A$. The *cosine* of an angle is, therefore, the *sine* of its *complement;* the *cotangent* of an angle the *tangent* of its *complement;* and the *cosecant* of an angle the *secant* of its *complement;* so that the six trigonometrical ratios of an acute angle are really composed of three ratios belonging to that angle, and three ratios belonging to its complement.

6. The Trigonometrical Ratios, *for each acute angle, are* CONSTANT; i. e., *are the same for the same* or equal angle, *in whatever right-angled triangle it is situated.*

Let $A$ represent any acute angle formed by the two lines $AE$ and $AF$, meeting at $A$. Make $ABC$ the triangle of reference. From $B$ draw $BD$ perpendicular to $AC$, and from $D$, where $BD$ meets $AC$, draw a perpendicular $DH$ to $AB$.



In triangle $ADH$, sin. $A = \frac{DH}{AD}$; in triangle $ABC$, sin. $A = \frac{BC}{AC}$; and in triangle $ABD$, sin. $A = \frac{BD}{AB}$ (1 Art. 4). But $\frac{DH}{AD} = \frac{BC}{AC}$(Euc. 4, VI. Ch. 4, III.) $= \frac{BD}{AB}$ (Euc. 8, VI. Ch. 13, III.).

The angle $A = DBC$ (Euc. 8, VI. Ch. 13, III.) $=$ $BDH$ (Euc. 29, I. Ch. 13, I.). Sin. $DBC = \frac{DC}{BC} = \frac{BC}{AC}$ $=$ sin. $A$. Also, sin. $BDH = \frac{HB}{BD} = \frac{BD}{AB} = \frac{BC}{AC}$ $=$ sin. $A$.

In triangle $ADH$, tan. $A = \frac{DH}{AH}$; in triangle $ABC$, tan. $A = \frac{BC}{AB}$; and in triangle $ABD$, tan. $A = \frac{BD}{AD}$. But $\frac{DH}{AH} = \frac{BC}{AB} = \frac{BD}{AD}$.

Also, tan. $DBC = \frac{DC}{DB} = \frac{BD}{AD} =$ tan. $A$; tan. $HDB = \frac{HB}{HD} = \frac{BD}{AD} = \frac{HD}{AH} =$ tan. $A$.

In triangle $ADH$, sec. $A = \frac{AD}{AH}$; in $ABC$, sec. $A$ $= \frac{AC}{AB}$; and, in $ABD$, sec. $A = \frac{AB}{AD}$. But $\frac{AD}{AH}$ $= \frac{AC}{AB} = \frac{AB}{AD}$.

Also, sec. $DBC = \frac{BC}{BD} = \frac{AC}{AB} =$ sec. $A$; and sec. $HDB = \frac{BD}{HD} = \frac{AB}{AD} =$ sec. $A$.

In a similar manner it may be proved that the ratios for the cosine, cotangent, and cosecant of the same or equal angle are constant.

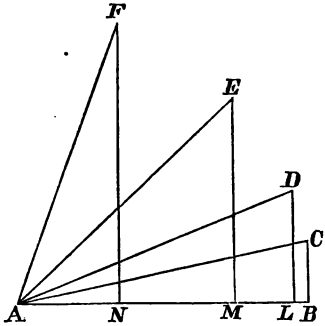
7. The SINE *and the* COSINE, *of an acute angle, are always* LESS THAN 1.

The *sine* being equal to the ratio of the perpendicular to the hypotenuse of the triangle of reference, in the fraction expressing that ratio the numerator is always less than the denominator, and therefore the value of the fraction is less than 1.

The *cosine* being equal to the ratio of the base to the hypotenuse, in the fraction expressing that ratio the numerator is always less than the denominator, and therefore the value of the fraction is less than 1.

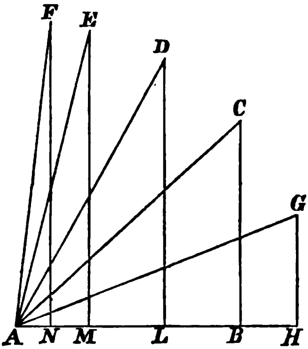
Thus let $ABC$, $ALD$, $AME$, and $ANF$, be a series of right-angled triangles, all having an hypotenuse of the same length. Let the hypotenuse of each triangle represent 32 units of length. Let $CB = 6$ units; $DL = 12$ units; $ME = 22$ units; $NF = 31$ units. The numerical measures of these lines are, therefore, 32, 6, 12, 22, etc.

Sin. $CAB = \frac{CB}{AC} =$ (1 Art. 4) $= \frac{6}{32}$ (Ch. Art. 43, II.); sin. $DAL = \frac{DL}{DA} = \frac{12}{32}$; sin. $EAM = \frac{EM}{AE} = \frac{22}{32}$; sin. $FAN = \frac{FN}{FA} = \frac{31}{32}$; all less than 1.



Also, let there be a series of right-angled triangles $GAH$, $ABC$, etc., all having an hypotenuse of the same length, viz., 16. Let $AH = 15$, $AB = 12$, $AL = 8$, $AM = 4$, and $AN = 2$, in terms of the unit of length.

Cos. $GAH = \frac{AH}{AG}$ (4, Art. 4) $= \frac{15}{16}$; cos. $CAB = \frac{AB}{AC} = \frac{12}{16}$; cos. $DAL = \frac{AL}{AD} = \frac{8}{16}$; etc., all less than 1.



Art. 8. Cor. As the angle increases, the sine increases, but the cosine decreases.

9. (Defs.) If the cosine of an angle be subtracted from unity, the remainder is called the VERSED SINE of the angle. Thus $1 - \cos. A = \text{versin.}\ A$.

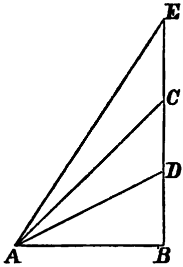
If the sine of an angle be subtracted from unity, the remainder is called the COVERSED SINE of the angle. Thus $1 - \sin. A = \text{coversin.}\ A$.

10. The TANGENT and the COTANGENT, of *an acute angle, may be* LESS THAN 1, EQUAL TO 1, or GREATER THAN 1.

The tangent being equal to the ratio of the perpendicular to the base of a right-angled triangle, the fraction expressing the ratio will be less than 1, when the perpendicular is less than the base; will be equal to 1, when the perpendicular is equal to the base; will be greater than 1, when the perpendicular is greater than the base.

The tangent of an angle will be less than 1, when the angle is less than 45°; will be equal to 1, when the angle is equal to 45°; will be greater than 1, when the angle is greater than 45°.

Thus let $ABD$, $ABC$, and $ABE$, be triangles right angled at $B$. Let the numerical measure of $AB$ and of $BC$ each equal 10; let the numerical measure of $BD$ equal 5, and of $BE$ equal 15; all referred to the same unit of length.



$ABC$ is an isosceles triangle, and the angles $BAC$ and $BCA$ are each equal to 45° (Euc. 5 and 32, I. Ch. 18 and 25, I.);

Therefore $DAB$ is less than 45°, and $EAB$ is greater than 45°.

Now, tan. $DAB = \frac{DB}{AB}$ (2 Art. 4) $= \frac{5}{10} = \frac{1}{2}$;

Therefore the tangent of an angle less than 45° is less than 1.

Tan. $CAB = \frac{CB}{AB} = \frac{10}{10} = 1$;

Therefore the tangent of an angle of 45° equals 1.

Tan. $EAB = \frac{EB}{AB} = \frac{15}{10} = 1\frac{1}{2}$;

As above, the tangent of an angle greater than 45° is greater than 1.

Again, the cotangent of an angle being equal to the ratio of the base to the perpendicular of a right-angled triangle, is less than 1, equal to 1, or greater than 1, according as the base is less than the perpendicular, equal to the perpendicular, or greater than the perpendicular; that is, according as the angle is greater than 45°, equal to 45°, or less than 45°.

Taking the construction of the above figure,

Cot. $EAB = \frac{AB}{BE}$ (4 Art. 4) $= \frac{10}{15} = \frac{2}{3}$;

Therefore the cotangent of an angle greater than 45° is less than 1.

Cot. $CAB = \frac{AB}{BC} = \frac{10}{10} = 1$;

Therefore the cotangent of 45° equals 1.

$$\text{Cot. } DAB = \frac{AB}{BD} = \frac{10}{5} = 2;$$

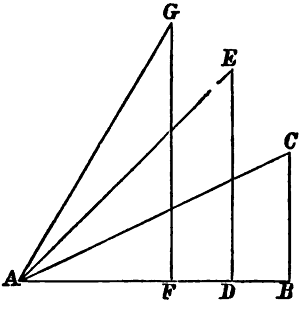
Therefore the cotangent of an angle less than 45° is greater than 1.

11. Cor. As the acute angle increases, its *tangent increases*, but its *cotangent decreases.*

12. The SECANT *and the* COSECANT, *of an acute angle, are* ALWAYS GREATER THAN 1.

The secant of an acute angle being equal to the ratio of the hypotenuse to the base of a right-angled triangle, the fraction expressing that ratio always has its numerator greater than its denominator, and is therefore always greater than 1.

Let $ABC$, $ADE$, and $AFG$, be triangles right angled at $B$, $D$, and $F$, respectively. Let the numerical measure of $AC$, $AE$, and $AG$, each equal 20; let the numerical measure of $AB$ equal 18, of $AD$ equal 14, and of $AF$ equal 10.



$$\text{Sec. } CAB = \frac{AC}{AB} = \frac{20}{18} = \frac{10}{9} = 1\tfrac{1}{9};$$

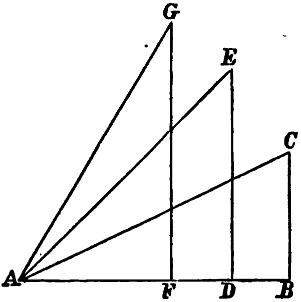
$$\text{Sec. } EAD = \frac{AE}{AD} = \frac{20}{14} = 1\tfrac{3}{7};$$

$$\text{Sec. } GAF = \frac{AG}{AF} = \frac{20}{10} = 2.$$

Each of these secants is greater than 1, and, as above, the secant of an acute angle is always greater than 1.

Again, the cosecant being equal to the ratio of the hypotenuse to the perpendicular, the fraction expressing that ratio is always greater than 1, as its numerator is always greater than its denominator.

Let $ABC$, $ADE$, and $AFG$, be triangles right angled at $B$, $D$, and $F$, respectively. Let the numerical measures of $AC$, $AE$, and $AG$, each equal 20; let the numerical measure of $BC$ equal 8, of $DE$ equal 14, and of $FG$ 16, respectively.



$$\text{Cosec. } CAB = \frac{AC}{CB} = \frac{20}{8} = 2\tfrac{1}{2};$$

$$\text{Cosec. } EAB = \frac{AE}{DE} = \frac{20}{14} = 1\tfrac{3}{7};$$

$$\text{Cosec. } GAF = \frac{AG}{FG} = \frac{20}{16} = 1\tfrac{1}{4}.$$

Each of these is greater than 1, and, therefore, the cosecant of an acute angle is always greater than 1.

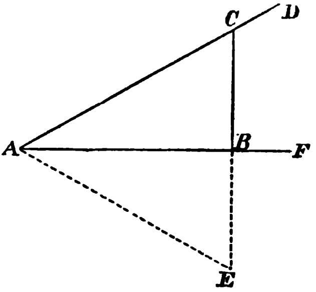
13. Cor. As the *acute* angle increases, its *secant increases*, but its *cosecant decreases.*

# CHAPTER III.

## TRIGONOMETRICAL RATIOS OF AN ANGLE OF 30°, OF 45°, AND OF 60°.

ART. 14. *To find the numerical values of the* TRIGONOMETRICAL RATIOS *of an angle of* 30°.

Suppose $DAF$ to be an angle of 30°. In $AD$ take any point, $C$, and from $C$ draw $CB$ perpendicular to $AF$, forming the triangle of reference, $CAB$. At the point $A$, in the line $AB$, make the angle $BAE$ equal to $BAC$—that is, to 30°—and produce $CB$ to meet $AE$ at $E$. Then $ABC$ and $ABE$ are equal triangles (Euc. 26, I. Ch. 21, I.), and $CB$ is equal to $BE$.



Now, in the triangle $ACE$, the angle $A$ is equal to 60°, and the angle $E$ is also equal to 60°. Therefore $CA$ is equal to $CE$ (Euc. 6, I. Ch. 27, I.), and $CB$, which is half of $CE$, is equal to half of $CA$.

Now, $AC^2 = AB^2 + BC^2$ (Euc. 47, I. Ch. 14, III.) $= AB^2 + \frac{1}{4}AC^2$;

Therefore $AB^2 = AC^2 - \frac{1}{4}AC^2 = \frac{3}{4}AC^2$; and $AB = \frac{\sqrt{3}}{2}AC$.

Now, sin. 30° $= \sin. CAB = \dfrac{CB}{AC} = \dfrac{CB}{2CB} = \dfrac{1}{2}$;

Tan. 30° $= \tan. CAB = \dfrac{CB}{AB} = \dfrac{\frac{1}{2}AC}{\frac{\sqrt{3}}{2}AC} = \dfrac{1}{\sqrt{3}}$

$= \frac{1}{3}\sqrt{3}$;

Sec. 30° $= \sec. CAB = \dfrac{AC}{AB} = \dfrac{AC}{\frac{\sqrt{3}}{2}AC} = \dfrac{2}{\sqrt{3}}$

$= \frac{2}{3}\sqrt{3}$;

Cos. 30° $= \cos. CAB = \dfrac{AB}{AC} = \dfrac{\frac{\sqrt{3}}{2}AC}{AC} = \frac{1}{2}\sqrt{3}$;

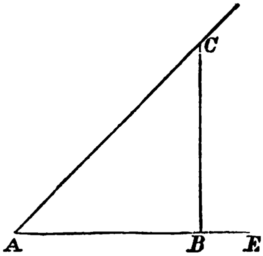
Cot. 30° $= \cot. CAB = \dfrac{AB}{BC} = \dfrac{\frac{\sqrt{3}}{2}AC}{\frac{1}{2}AC} = \sqrt{3}$;

Cosec. 30° $= \operatorname{cosec}. CAB = \dfrac{AC}{BC} = \dfrac{AC}{\frac{1}{2}AC} = 2$;

Versin. 30° $= \operatorname{versin}. CAB = 1 - \cos. 30°$ (Art. 9)

$= 1 - \dfrac{\sqrt{3}}{2}$.

15. *To find the numerical values of the* TRIGONOMETRICAL RATIOS *of an angle of* 45°.



Suppose $CAE$ to be an angle of 45°. In $AC$ take any point, $C$, and from $C$ draw $CB$ perpendicular to $AE$, meeting $AE$ at $B$.

In the right-angled triangle $ABC$, as the angle $A$ is equal to 45°, the angle $C$ is also equal to 45° (Euc. 32, I. Ch. 18, I.). Consequently $C$ is equal to $A$, and the side $AB$ to $BC$ (Euc. 6, I. Ch. 27, I.).

Now, $AC^2 = AB^2 + BC^2 = 2\overline{BC}^2 = 2\overline{AB}^2$;

therefore $AB = \dfrac{AC}{\sqrt{2}} = BC$.

Now, $\sin. 45° = \sin. A = \dfrac{BC}{AC} = \dfrac{\frac{1}{\sqrt{2}} AC}{AC} = \dfrac{1}{\sqrt{2}}$ $= \frac{1}{2}\sqrt{2}$;

$$\text{Tan. } 45° = \tan. A = \frac{BC}{AB} = \frac{BC}{BC} = 1;$$

$$\text{Sec. } 45° = \sec. A = \frac{AC}{AB} = \frac{AC}{\frac{1}{\sqrt{2}} AC} = \sqrt{2};$$

$$\text{Cos. } 45° = \cos. A = \frac{AB}{AC} = \frac{\frac{1}{\sqrt{2}} AC}{AC} = \frac{1}{\sqrt{2}} = \tfrac{1}{2}\sqrt{2};$$

$$\text{Cot. } 45° = \cot. A = \frac{AB}{BC} = \frac{AB}{AB} = 1;$$

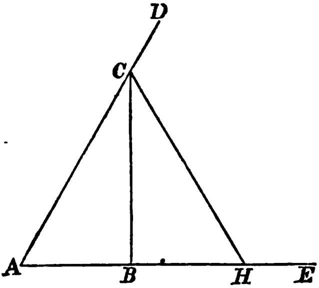
$$\text{Cosec. } 45° = \text{cosec. } A = \frac{AC}{BC} = \frac{AC}{\frac{1}{\sqrt{2}} AC} = \sqrt{2};$$

$$\text{Versin. } 45° = \text{versin. } A = 1 - \frac{1}{\sqrt{2}} = 1 - \tfrac{1}{2}\sqrt{2}.$$

16. *To find the numerical values of the* TRIGONOMETRICAL RATIOS *of an angle of* 60°.

Suppose $DAE$ to be an angle of 60°. In $AD$ take any point, $C$, and from $C$ draw $CB$ perpendicular to $AE$, meeting $AE$ in $B$. From $B$ lay off $BH$, equal to $AB$, and join $CH$.

Then the triangles $CAB$ and $BCH$ are equal (Euc. 4, I. Ch. 20, I.), and the angle $CHA$ is equal to $CAH$—that is, to 60°. Therefore the angle $ACH$ is also equal to 60°. In the triangle $ACH$ the side $AH$ is equal to $AC$, and $AB$, which is one-half of $AH$, is equal to one-half of $AC$.



Now, $AC^2 = AB^2 + BC^2$; therefore $BC^2 = AC^2 - AB^2 = AC^2 - \frac{1}{4}AC^2 = \frac{3}{4}AC^2$ and $BC = \frac{\sqrt{3}}{2}AC$.

$$\text{Now, sin. } 60° = \text{sin. } A = \frac{BC}{AC} = \frac{\frac{\sqrt{3}}{2}AC}{AC} = \tfrac{1}{2}\sqrt{3};$$

$$\text{Tan. } 60° = \text{tan. } A = \frac{BC}{AB} = \frac{\frac{\sqrt{3}}{2}AC}{\frac{1}{2}AC} = \sqrt{3};$$

$$\text{Sec. } 60° = \text{sec. } A = \frac{AC}{AB} = \frac{AC}{\frac{1}{2}AC} = 2;$$

$$\text{Cos. } 60° = \text{cos. } A = \frac{AB}{AC} = \frac{\frac{1}{2}AC}{AC} = \tfrac{1}{2};$$

$$\text{Cot. } 60° = \text{cot. } A = \frac{AB}{BC} = \frac{\frac{1}{2}AC}{\frac{\sqrt{3}}{2}AC} = \frac{1}{\sqrt{3}} = \tfrac{1}{3}\sqrt{3};$$

Cosec. $60^\circ$ = cosec. $A = \dfrac{AC}{BC} = \dfrac{AC}{\frac{\sqrt{3}}{2}AC} = \dfrac{2}{\sqrt{3}} =$
$\frac{2}{3}\sqrt{3}$;

Versin. $60^\circ$ = versin. $A = 1 - \frac{1}{2} = \frac{1}{2}$.

We might also derive the trigonometrical ratios of an angle of 60° from those of an angle of 30°, according to Art. 5.

Thus, sin. $60^\circ$ = cos. $30^\circ = \frac{1}{2}\sqrt{3}$;
Tan. $60^\circ$ = cot. $30^\circ = \sqrt{3}$;
Sec. $60^\circ$ = cosec. $30^\circ = 2$;
Cos. $60^\circ$ = sin. $30^\circ = \frac{1}{2}$;
Cot. $60^\circ$ = tan. $30^\circ = \frac{1}{3}\sqrt{3}$;
Cosec. $60^\circ$ = sec. $30^\circ = \frac{2}{3}\cdot\sqrt{3}$;
Versin. $60^\circ = 1 -$ cos. $60^\circ = 1 -$ sin. $30^\circ = \frac{1}{2}$.

EXAMPLE 1. If the hypotenuse of a right-angled triangle be 5, and the perpendicular be 4, required the trigonometrical ratios for the angle at the base. Required also the trigonometrical ratios for the angle at the perpendicular.

2. If the base and perpendicular of a right-angled triangle be 7 and 8, find the trigonometrical ratios of the angles at the base and perpendicular.

3. Calculate to four decimal places the numerical values of the trigonometrical ratios of an angle of 30°, and of 60°.

4. Calculate to four decimal places the numerical values of the trigonometrical ratios of an angle of 45°.

# CHAPTER IV.

## THE USE OF TRIGONOMETRICAL TABLES.—SOLUTION OF RIGHT-ANGLED TRIANGLES.

Art. 17. The *trigonometric ratios* for all angles between 0° and 90°, beginning at an angle of 1′ and increasing in size by successive additions of 1′, have been calculated and have been arranged in tables. Tables have also been calculated for angles beginning at an angle of 10″, and increasing in size by successive additions of 10″. Smaller tables, in which the interval between angles is 15′, have also been arranged for calculations where great accuracy is not required.

Such tables are called in general *Trigonometrical Tables*, and also "Tables of Natural Sines and Cosines," "Tables of Natural Tangents and Cotangents," etc.

18. The *logarithmic* values of the trigonometric ratios have also been arranged in tables, called "Tables of Logarithmic Sines," etc., for use in calculations by logarithms.

19. The values of ratios, intermediate between the ratios of the tables, are obtained from those of the tables on the theory that "*for small intervals, the differences of the ratios are proportional to the differences of the angles.*"

Thus suppose we are using a table of natural sines, in which the sines are calculated for intervals of 1′, and we are required to find the sine of an angle of 10° 1′ 1″. This sine falls between the sine of 10° 1′ and the sine of 10° 2′. In the table of sines, under 10° and opposite 1 in the column of Min. (minutes), we find the sine of 10° 1′ to be .173935, and, directly under it, and opposite 2 in the column headed Min., we find the sine of 10° 2′ to be .174221. (The decimal point is not prefixed in the table, but is always understood.) The difference of the two sines is $\frac{286}{1,000,000}$, or .000286, and the angles differ by 1′ or 60″. Therefore since when the angle increases 60″ the sine increases .000286, on the principle just enunciated, when the *angle* increases 1″, or $\frac{1}{60}$th of the former increase, the *sine* will also increase $\frac{1}{60}$th of its former increase, or will increase $\frac{.000286}{60} = \frac{47}{10,000,000} = .000005$, nearly. So that the sine of 10° 1′ 1″ is obtained by adding .000005 to .173935. The sine of 10° 1′ 1″ is therefore .173940.

If we wished to obtain the sine of 10° 1′ 2″ we should add twice the increase for one second, or $\frac{2}{60}$ths of the increase for 1′—that is, .000009, nearly—and so on.

The increase for 1″ is generally calculated at short intervals, and put in the table under the column of the ratios to which it belongs, with the name of "*proportional parts*." To find the increase for any number of seconds, we multiply the proportional parts for 1″ by the number of seconds.

To obtain from the tables a ratio for an angle between two angles of the table, it is best in general to take the ratio belonging to the *smaller* of the two angles, between which it falls, and apply the correction according to the principle above given, or directly from the "Corrections" or "Proportional Parts" of such tables, being careful to *add such correction, according as the ratio required is a sine, tangent, or secant;* and to *subtract the correction, if the required ratio is a cosine, cotangent, or cosecant* (Arts. 8, 11, 13).

20. Conversely, *to obtain the degrees, minutes, and seconds, answering to a given ratio intermediate between two ratios of the table, take the degrees and minutes belonging to the smaller of the two angles*, between whose ratios it falls, *and, for the seconds, divide the difference between this ratio and the given ratio by the proportional parts for* 1″, *found under the column in which the two ratios appear.*

Thus suppose we are using a five-place table, and we have a sine given as .50060, and are required to find the number of degrees, minutes, and seconds, in the angle to which it belongs.

Looking in the table of sines we find it falls between .50050 and .50076; that is, that it belongs to an angle between 30° 2′ and 30° 3′. The difference between the sine belonging to the smaller angle and the given sine is 10—that is, $\frac{10}{100{,}000}$—and the proportional part for 1″ is $\frac{26}{60} = 0.43$ (considering 26 as a whole number). Dividing 10 by .43 we have for the seconds, 23. Therefore the angle whose sine is .50060, is an angle of 30° 2′ 23″.

21. The logarithm of a ratio intermediate between the logarithms of two ratios is obtained on the same general principle as were the ratios themselves, it being assumed that, for small intervals, the difference of the logarithms of the ratios is proportional to the difference of the angles.

The principle here assumed is not strictly true, nor is the principle assumed in the previous article strictly true; but, employing it with the limitation *of small intervals*, we are not in danger of error, except in certain angles near 0°, and near 90°, for which angles separate tables are provided.

22. In the tables of logarithmic sines, cosines, tan-

gents, etc., the logarithms are generally calculated for intervals of ten seconds, and the corrections or proportional parts are given for the seconds from 1″ to 9″.

Most of the trigonometric ratios, whose logarithms are given, are less than 1 (Arts. 7 and 10), and, therefore, the characteristics of their logarithms would be negative (Art. 404, Loomis's Alg.). To avoid the use of negative characteristics, however, 10 is always added to the logarithm. Consequently, in order to obtain the correct logarithm of a result, *in a calculation in which trigonometric ratios have been represented by their logarithms*, from the resulting logarithm *10 must be subtracted for every trigonometric ratio used as a multiplier*, and, to the resulting logarithm, *10 must be added for every trigonometric ratio used as a divisor.*

## SOLUTION OF RIGHT-ANGLED TRIANGLES.

Art. 23. The *parts* of any triangle are the sides and the angles.

To *solve* a triangle is to find the unknown parts from certain known parts.

*Trigonometry* was, primarily, the science by which triangles are solved, and was originally so defined.

24. The parts of *any* triangle are six in number. In a *right-angled* triangle, as one of the parts is always a right angle, *one* part is always known. As the two acute angles together make a right angle, when *one* acute angle is known, the *other*, being its complement, is also known. So that to *solve* a *right-angled* triangle it is only necessary to consider *four parts;* viz., the

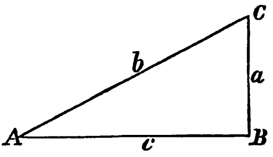
*three sides and one acute angle. Any two of these parts of a right-angled triangle being known, we are able, by the use of trigonometric tables, to solve the triangle.*

25. Suppose two sides are known. These may be the *hypotenuse* and *perpendicular;* the *hypotenuse* and *base;* or the *perpendicular* and *base.*

As the perpendicular is "the side opposite the given angle" (Art. 4), either of the sides about the right angle may be made the perpendicular, and the other side the base, or "the side adjacent to the given angle," according as we take one or the other of the acute angles as the given angle. When two sides are given we really have, therefore, two cases only, viz.: 1. When the hypotenuse and a side are given; 2. When the two sides about the right angle are given.

26. *The hypotenuse and a side being known, to solve the triangle.*

Let $A B C$ be a right-angled triangle, having its right angle at $B$. Denote the sides opposite the angle by small letters of the same name as the capital letters denoting the angles.



(Generally, in all triangles, the notation will be adopted of capitals for the angles, and small letters of the same name for the opposite sides.)

In the triangle $A B C$ suppose we have given the hypotenuse $b$, and the side $a$.

(1) $\text{Sin. } A = \frac{a}{b}$ (1 Art. 4).

(2) Also $\frac{c}{b} = \text{cos. } A$ (4 Art. 4); or, $c = b \text{ cos. } A$

(3) $c$ also equals $\sqrt{b^2 - a^2} = \sqrt{(b + a)(b - a)}$; a formula convenient for the use of logarithms, as then $\log. c = \frac{1}{2} \log. \{(b + a) + \log. (b - a).\}$

From (1) we see that, when the hypotenuse and a side are given, *the angle opposite the given side is an angle whose sine is equal to the ratio of the side to the hypotenuse.* It is, therefore, to be found from the table of sines.

When this angle is found, the third side can be found by formula (2).

27. If the hypotenuse, $b$, and the side, $c$, were given, we could find the angle $C$, as we found the angle $A$; or, if we desired to find the angle $A$, we have

$$\frac{c}{b} = \cos. A;$$

that is, when the hypotenuse and a side are given, the angle *adjacent to the given side is an angle whose cosine is the ratio of the given side to the hypotenuse.* It is, therefore, to be found from a table of cosines.

In this case the other side, $a = b \sin. A$;

$$\text{for } \frac{a}{b} = \sin. A; \therefore a = b \sin. A;$$

$$\text{or, } a = \sqrt{b^2 - c^2} = \sqrt{(b + c)(b - c)}.$$

EXAMPLE 1. If the hypotenuse and perpendicular of a right-angled triangle be 192 and 130, respectively, required the other parts of the triangle. *Ans.* Angles = 47° 23′ 2″ and 42° 36′ 58″; side = 142.29.

2. The hypotenuse and base of a right-angled triangle being 71 and 64, respectively, required the other parts of the triangle.

*Ans.* Angles = 25° 39′ 22″ and 64° 20′ 38″; side = 30.74.

3. If the hypotenuse of a right-angled triangle be 71, and the base 60, required the other parts.

*Ans.* Angles = 32° 19′ 14″ and 57° 40′ 46″; side = 37.96.

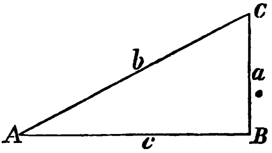
4. If the hypotenuse and a side of a right-angled triangle be 140 and 84, respectively, what are the other parts?

5. If the hypotenuse and a side of a right-angled triangle be 130 and 66, respectively, what are the other parts?

6. The hypotenuse and perpendicular of a right-angled triangle are 200 and 100, respectively; what are the other parts?

Art. 28. *The two sides about the right angle being known, to solve the triangle.*

In the figure, suppose the two sides, $a$ and $c$, are known, and it is required to find the hypotenuse and angles.



$b = \sqrt{a^2 + c^2}$ (Euc. 47, I. Ch. 14, III.).

(1) Tan. $A = \frac{a}{c}$ (2 Art. 4).

After the angle $A$ is found, we can also find the hypotenuse, thus:

(2) $\frac{a}{b} = \text{sin. } A$; $\therefore\ b = \frac{a}{\text{sin. } A}$; or,

(3) $\frac{b}{c} = \text{sec. } A$; $\therefore\ b = c \text{ sec. } A$.

From (1) we see that *when the two sides of a right-angled triangle are given, the angle opposite either of the sides is an angle whose tangent is the ratio of that side to the other side.* It is, therefore, to be found from a table of tangents.

EXAMPLE 1. If the base of a right-angled triangle be 141, and the perpendicular be 193, required the angles and the hypotenuse.

*Ans.* 53° 50′ 57″, 36° 9′ 3″; 239.02, nearly.

**2.** If the two sides about the right angle of a right-angled triangle be 2.1 and 2, what are the angles and the hypotenuse?

*Ans.* 43° 36′ 10″, 46° 23′ 50″; 2.9.

**3.** If the two sides of a right-angled triangle be 123.5 and 10.16, what are the angles and the hypotenuse?

*Ans.* 4° 42′ 11″, 85° 17′ 49″; 123.92.

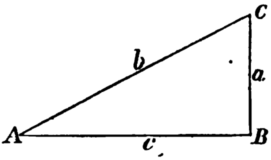
**4.** If the two sides of a right-angled triangle be 39 and 30, what are the other parts?

**5.** If the two sides of a right-angled triangle be 81.48 and 108.64, what are the other parts?

**6.** If the two sides of a right-angled triangle are 131 and 13.1, find the other parts.

**7.** If the sides of a right-angled triangle are each equal to *a*, what is the hypotenuse and what are the angles?

29. Now, suppose a side and an acute angle are known. These may be either the hypotenuse and an acute angle, or one of the sides about the right angle and an acute angle.



30. *The hypotenuse and an acute angle being known, to solve the triangle.*

In the right-angled triangle, $ABC$, suppose the hypotenuse $b$, and the angle $A$, are known.

$$(1)\ \text{Sin. } A = \frac{a}{b};\ \therefore\ a = b \sin. A.$$

$$(2)\ \text{Cos. } A = \frac{c}{b};\ \therefore\ c = b \cos. A.$$

In the same triangle suppose the hypotenuse $b$, and the angle $C$, are known.

$$(3)\ \text{Sin. } C = \frac{c}{b};\ \therefore\ c = b \sin. C.$$

$$(4)\ \text{Cos. } C = \frac{a}{b};\ \therefore\ a = b \cos. C.$$

From (1) and (3) it will be seen that, in a right-angled triangle, *the side opposite an acute angle is equal to the product of the hypotenuse by the sine of the angle.*

From (2) and (4) it will be seen that, in a right-angled triangle, *the side adjacent to an acute angle is equal to the product of the hypotenuse by the cosine of the angle.*

Example 1. If the hypotenuse of a right-angled triangle be 4.958, and one of the angles be 54° 44′, find the other sides.

*Ans.* 4.048, 2.8626.

2. The hypotenuse of a right-angled triangle is 25, and one of the acute angles is 16° 15′ 37″. Required the sides. *Ans.* 7 and 24.

3. The hypotenuse of a right-angled triangle is 37.36, and one of the acute angles is 12° 30′. Required the sides. *Ans.* 8.0862, 36.474.

4. The hypotenuse of a right-angled triangle is 120, and one of the acute angles is 36° 14′ 15″. Required the sides.

*Ans.* 70.936, 96.789.

5. The hypotenuse of a right-angled triangle is 316.5, and one of the acute angles is 55° 30′ 17″. Required the sides.

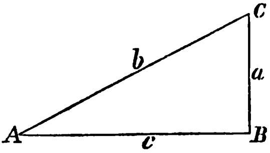
6. The hypotenuse of a right-angled triangle is 656.15, and one of the acute angles is 75° 10′ 5″. Required the sides.

7. The hypotenuse of a right-angled triangle is 100, and one of the acute angles is 60°. Required the sides.

8. The hypotenuse of a right-angled triangle is 567, and one of the acute angles is 45°. Required the sides.

9. The diagonal of a square is 400. Required the length of a side.

10. The side of a rhombus is 48 feet long, and one of its angles is 68°. Required the length of its diagonals.



31. *One of the sides about the right angle and one of the acute angles being known, to solve the triangle.*

In the right-angled triangle $A\,B\,C$, suppose the side $a$, and the angle $A$, opposite the side $a$, are known.

(1) Sin. $A = \frac{a}{b}$; $\therefore\ b = \frac{a}{\sin. A}$.

(2) Tan. $A = \frac{a}{c}$; $\therefore\ c = \frac{a}{\tan. A}$.

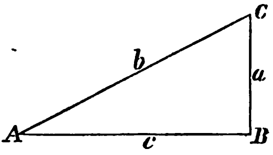
(3) Or cosec. $A = \frac{b}{a}$; $\therefore\ b = a$ cosec. $A$.

(4) Cot. $A = \frac{c}{a}$; $\therefore\ c = a$ cot. $A$.

Formula (1) and formula (2) are more convenient for general use than formula (3) and formula (4).

From (1) and (2) it will be seen that when a side about the right angle of a right-angled triangle, and an angle opposite it, are given, *the hypotenuse is equal to the quotient of the given side divided by the sine of the given angle, and the other side is equal to the quotient of the given side divided by the tangent of the given angle.*

32. If, instead of a side and the opposite angle being known, we know a side and the adjacent angle, we can find the opposite angle—as each acute angle is the complement of the other—and then we can solve the triangle, as in Art. 31. Or we can solve the triangle directly, thus:

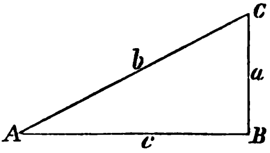


Suppose, in the right-angled triangle $A\,B\,C$, the side $c$ and the angle $A$ are given, to solve the triangle.

(1) Cos. $A = \frac{c}{b}$; $\therefore\ b = \frac{c}{\cos. A}$.

(2) Tan. $A = \frac{a}{c}$; $\therefore$ $a = c$ tan. $A$.

(3) Also, sec. $A = \frac{b}{c}$; $\therefore$ $b = c$ sec. $A$.



From (2) and (3) it will be seen that, in a right-angled triangle, *the side opposite an acute angle is equal to the product of the side adjacent to that angle by the tangent of the angle;* and that *the hypotenuse is equal to the product of a side by the secant of the angle adjacent to the side.*

Example 1. The side of a right-angled triangle is 141, and the angle opposite this side is 33° 41′ 6″. Required the hypotenuse and the other side. *Ans.* 254.225, 211.54.

2. The side of a right-angled triangle is 124.6, and the angle opposite it is 64° 20′. What is the hypotenuse and the other side? *Ans.* 138.24, 59.8766.

3. The side of a right-angled triangle is 19.67, and the angle opposite to it is 18° 31′ 4″. Required the hypotenuse and the other side.

4. The side of a right-angled triangle is 111.11, and the angle adjacent to this side is 73° 49′. Required the hypotenuse and the other side.

5. If one side of a rectangle is 50 feet, and the diagonal makes an angle of 30° with this side, what is the length of the diagonal, and what is the length of the other sides of the rectangle?

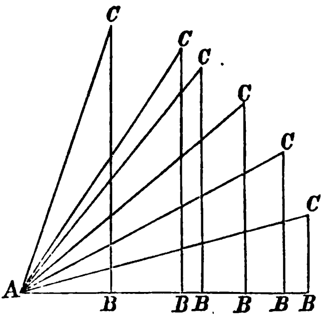
6. The diagonal of a rhombus is 30 feet, and the angle through which the diagonal passes is 120°. What is the length of a side of the rhombus, and what is the length of the other diagonal?

33. From the preceding articles it will be seen that to solve a right-angled triangle we simply apply the definitions of trigonometric ratios. To find a required part we select, in each case, the definition in which occur the required part and the two known parts.

# CHAPTER V.

## TRIGONOMETRIC RATIOS OF A RIGHT ANGLE, OF 0, AND OF OBTUSE ANGLES.

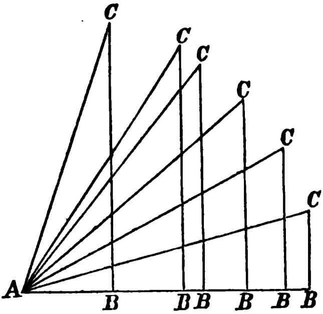
ART. 34. According to Art. 3, the *triangle of reference* for the trigonometric ratios of an angle is formed by dropping a perpendicular upon one side, or side produced containing the angle, from any point in the other side. In the case of a right angle, the perpendicular coincides with one of the sides of the angle and no triangle is formed. We shall, therefore, obtain the trigonometric ratios of a right angle by the method of limits, considering the case of a right-angled triangle, whose hypotenuse remains constant, while one of the acute angles, taken as a variable, "approaches indefinitely" to a right angle as its limit (Ch. Art. 28, Bk. V.).



Art. 35. *The sine of a right angle, or the sine of an angle of 90° is 1.*

Let $A\,B\,C$ be a triangle right-angled at $B$. Suppose, while the hypotenuse remains the same, the acute angle $B\,A\,C$ increases. At the same time $B\,C$ increases. Also the sine of $B\,A\,C$ increases (Art. 8).

Now, as the angle $BAC$ increases, it approaches a right angle as its limit, and its sine approaches the sine of a right angle as its limit. Also as the angle $BAC$ increases and approaches a right angle as its limit, the perpendicular, $BC$, approaches the hypotenuse, $AC$, as its limit (Euc. 6, I. Ch. 27, I.); and, therefore, $\frac{BC}{AC}$ approaches $\frac{AC}{AC}$, or 1, as its limit. But sin. $BAC$ always equals $\frac{BC}{AC}$; therefore, at the limits, sin. 90°, or the sine of a right angle, equals 1 (Ch. Art. 29, Bk. V.).



36. *The cosine of a right angle, or the cosine of an angle of 90°, is 0.*

When the angle $A$ (*see* figure) increases toward its limit (a right angle), its cosine decreases and approaches the cosine of a right angle as its limit (Art. 8). At the same time that $A$ increases, $AB$ decreases, and approaches 0 as its limit, so that the limit of $\frac{AB}{AC}$ is $\frac{0}{AC}$, or 0. But cos. $A$ always equals $\frac{AB}{AC}$; therefore, at the limits, the cosine of a right angle, or cos. 90°, equals 0.

37. *The tangent of a right angle, or the tangent of an angle of 90°, equals* $\infty$.

When the angle $A$ increases (*see* figure, page 40), the tangent increases (Art. 11). When the angle $A$ approaches a right angle as its limit, its tangent approaches the tangent of a right angle as its limit. Also as the angle $A$ increases, at the same time $CB$ increases toward $AC$, as its limit, while $AB$ decreases toward 0 as its limit, so that $\frac{BC}{AB}$ approaches $\frac{AC}{0}$ or $\infty$ as a limit. But $\tan. A = \frac{BC}{AB}$; therefore, at the limits, the tangent of a right angle, or tangent of 90°, equals $\infty$.

38. By a similar method of proof, it can be shown that *the cotangent of a right angle is 0;* that *the secant of a right angle is* $\infty$; and that *the cosecant of a right angle is 1.*

39. *To find the trigonometric ratios of 0°.* As 0° is the complement of 90°, we can determine the trigonometric ratios of 0° from those of a right angle, according to Art. 5.

Thus $\sin. 0° = \cos. 90° = 0$;
$\text{Cos. } 0° = \sin. 90° = 1$;
$\text{Tan. } 0° = \cot. 90° = 0$;
$\text{Cot. } 0° = \tan. 90° = \infty$;
$\text{Sec. } 0° = \text{cosec. } 90° = 1$;
$\text{Cosec. } 0° = \sec. 90° = \infty$.

40. We can also find the trigonometric ratios of 0° by the method of limits, considering 0° as the limit to which an acute angle approaches indefinitely when decreasing.

Thus, take the case of the sine of 0°.

When $A$ approaches 0° (*see* figure, page 40), as its limit, sin. $A$ approaches sin. 0° as its limit, and $CB$ approaches 0 as its limit. Therefore the limit of $\frac{CB}{AC} = \frac{0}{AC} = 0$. But sin. A always equals $\frac{CB}{AC}$. Therefore, at the limits, we shall have sin. 0° $= 0$ (Ch. Bk. V., Art. 29).

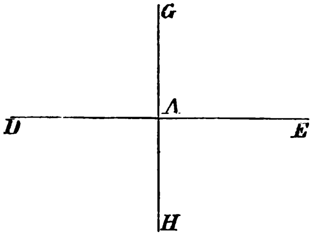
In a similar manner the other trigonometric ratios of 0 can be found.

### TRIGONOMETRICAL RATIOS OF AN OBTUSE ANGLE.

Art. 41. Before proceeding to the trigonometrical ratios of an obtuse angle, we will define the *sign* of a line as distinguished from its *value*, or numerical measure.

The *sign* of a line generally denotes the *direction* in which it is measured. To lines measured in one direction there are given positive signs (not always prefixed), while to lines measured in the opposite direction there are given negative signs.

Thus, taking two lines, $DE$ and $GH$, which we will name *initial lines*, intersecting at right angles at A, we call all lines going to the right of $GH$, or from $A$ toward $E$, *positive*, and we call all lines to the left of $GH$, or going from $A$ toward $D$, *negative*. So we call all perpendiculars upon $DE$, from above $DE$, *positive* lines, and we call all perpendiculars upon $DE$, from below, *negative*.

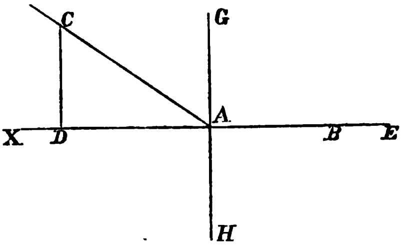


We shall also consider the hypotenuse of a right-angled triangle always positive.

Now, if we put the vertex of the angle, whose trigonometric ratios we are to consider, at $A$, and one of its sides coincident with $A\,E$, it will be apparent that (in the sense of our definitions with regard to the signs of lines), the *trigonometric ratios of all angles thus far considered are positive*, because the sides of the triangle of reference are all positive.

42. Take now an obtuse angle, and apply its vertex to the point $A$ (see figure), and let one of its sides, as $A\,B$, be coincident with $A\,E$.

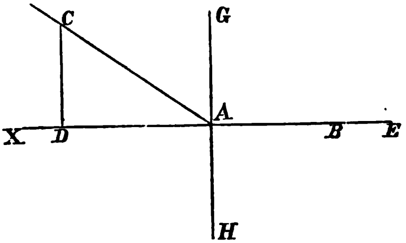
To construct the *triangle of reference* (Art. 3), we drop a perpendicular, $C\,D$, from any point, $C$, in $A\,C$, upon $A\,B$ produced, meeting $A\,B$ produced in $D$. In this triangle, $A\,D$, the side adjacent to the obtuse angle is negative, because we consider all lines running from $A$, toward the right of $G\,H$, as positive; and those from $A$, toward the left of $G\,II$, *negative* (Art. 41).



43. The trigonometrical ratios of an obtuse angle have the same *names* as the corresponding ratios of an acute angle, but have not always the same sign. For, as $A\,D$, the *base* of the triangle of reference, or the side *adjacent* to the obtuse angle, is always negative, and as the hypotenuse, $A\,C$, and the perpendicular,

$CD$, are always positive (Art. 41), every ratio in which $AD$ occurs must be negative.

44. Thus (1), the *sine* of an *obtuse angle* is the ratio of the *perpendicular* to the *hypotenuse* of the triangle of reference, and is positive (Art. 41).



$$\text{Sin. } CAB = \frac{CD}{AC} = \text{positive quantity.}$$

(2.) The *tangent* of an *obtuse angle* is the ratio of the *perpendicular* to the *base*, and is *negative.*

$$\text{Tan. } CAB = \frac{CD}{AD} = \frac{\text{positive quantity}}{\text{negative quantity}} = \left\{ \begin{matrix}\text{negative}\\ \text{quantity.}\end{matrix} \right.$$

(3.) The *secant* of an *obtuse angle* is the ratio of the *hypotenuse* to the *base*, and is *negative.*

$$\text{Sec. } CAB = \frac{AC}{AD} = \frac{\text{positive quantity}}{\text{negative quantity}} = \left\{ \begin{matrix}\text{negative}\\ \text{quantity.}\end{matrix} \right.$$

(4.) The *cosine* of an *obtuse angle* is the ratio of the *base* to the *hypotenuse*, and is *negative.*

$$\text{Cos. } CAB = \frac{AD}{AC} = \text{negative quantity.}$$

(5.) The *cotangent* of an *obtuse angle* is the ratio of the *base* to the *perpendicular*, and is *negative.*

$$\text{Cot. } CAB = \frac{AD}{DC} = \text{negative quantity.}$$

(6.) The *cosecant* of an *obtuse angle* is the ratio of the *hypotenuse* to the *perpendicular* of the triangle of reference, and is *positive.*

Cosec. $CAB = \frac{AC}{CD} =$ positive quantity.

45. It can be proved that the *sine and cosine of an obtuse angle* are always *less than 1* in numerical value, in the same manner that the sine and cosine of an acute angle have been proved less than 1 (Art. 7); also, that *when an obtuse angle increases* its *sine decreases*, but that its *cosine increases* (negatively) in numerical value.

It may also be proved that *the tangent* and *cotangent of an obtuse angle may be equal to* $-1$, *greater than* $-1$, *or less than* $-1$, as it was proved that the tangent and cotangent of an acute angle might be equal to 1, greater than 1, or less than 1 (Art. 10); also, that as the *obtuse angle increases* its *tangent decreases*, but its *cotangent increases*, both *negatively*.

It may further be proved that the *secant* and *cosecant* of an obtuse angle are *always greater than 1* numerically (the *secant* being greater than $-1$, the cosecant being greater than $+1$), in the same manner that the secant and cosecant of an acute angle were proved to be always greater than 1 (Art. 12); also, that *as the obtuse angle increases*, the secant decreases (*negatively*), but its cosecant increases.

Def. The versed sine of an obtuse angle is equal to the *algebraic* difference between 1 and the cosine of the angle.

Thus (*see* figure, page 43), versin. $A = 1 - \cos. A$. But, as cos. $A$ is a negative quantity (4, Art. 44), versin. $A$ is greater than 1. (In the case of an acute angle, the versed sine is less than 1.)

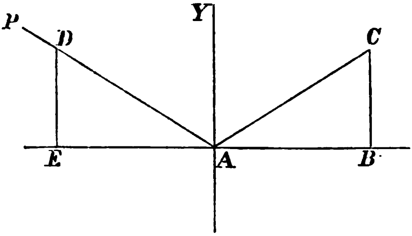
46. Any *trigonometrical ratio of an acute or obtuse*

*angle has the same numerical value as the corresponding ratio of its supplement.*

*The* SINE *and* COSECANT *of an acute angle, and the* SINE *and* COSECANT *of its supplementary obtuse angle, are all of the same sign, and are positive.*

*The* TANGENT, COTANGENT, COSINE, *and* SECANT *of an acute angle are positive, while the corresponding ratios of the supplementary obtuse angle are negative.*

In the figure $CAB$ is an acute angle. Produce $BA$ through $A$, beyond the perpendicular $AY$. At $A$, in the straight line $AE$, make the angle $PAE$ equal to $CAB$. Then is $PAB$ the supplement of $PAE$ (Ch. Bk. I., Art. 19), and, consequently, equal to the supplement of $CAB$, the equal of $DAE$.



Construct the triangle of reference, $CAB$. In $AP$ take $AD$ equal to $AC$, and from $D$ draw $DE$ perpendicular to $AE$. Then is $DAE$, the triangle of reference for the obtuse angle $DAB$, equal to the triangle $CAB$ (Euc. 26, I. Ch. 23, I.).

In the triangles $ACB$ and $ADE$ the hypotenuses $AC$ and $AD$, and the perpendiculars $CB$ and $DE$, and the base, $AB$, are all positive, while the base, $AE$, is negative (Arts. 41 and 42).

As the triangles of reference are equal in all respects, and therefore similar, the ratio between any two of the sides of one is equal to the ratio between the corresponding sides of the other (Euc. 4, VI. Ch. 4, III.); that is, the *numerical* value of any trigonometric ratio of the acute angle, $CAB$, is equal to the *numerical* value of the corresponding ratio of the obtuse angle, $DAB$.

Now, sin. $CAB = \frac{CB}{AC}$; also, sin. $DAB = \frac{DE}{DA}$. But $\frac{CB}{AC} = \frac{DE}{DA}$, as the triangles $CAB$ and $DAE$ are equal. Also as the lines $CB$, $DE$, $AC$, and $AD$, are all positive, the ratios $\frac{CB}{AC}$ and $\frac{DE}{DA}$ are positive, and therefore the sine of the angle, $CAB$, and the sine of its supplement, $DAB$, are of the same sign, and are positive.

Again, cosec. $CAB = \frac{AC}{CB}$; also, cosec. $DAB = \frac{DA}{DE}$. But $\frac{AC}{CB} = \frac{DA}{DE}$, and, as the terms of these ratios are *positive*, the ratios are positive; and, therefore, the cosecant of an angle and the cosecant of its supplement are of the same sign, and are *positive.*

Tan. $CAB = \frac{CB}{AB} =$ a positive quantity, as the lines $CB$ and $AB$ are both positive. But tan. $DAB$ (the supplement of $CAB$) $= \frac{DE}{AE} =$ a negative quantity, as $AE$ is negative (Arts. 41 and 42).

Cotan. $CAB = \frac{AB}{BC} =$ a positive quantity;

Cotan. $DAB = \frac{AE}{DE} =$ a negative quantity.

$$\text{Cos. } CAB = \frac{AB}{AC} = \text{a positive quantity;}$$

$$\text{Cos. } DAB = \frac{AE}{AD} = \text{a negative quantity.}$$

$$\text{Sec. } CAB = \frac{AC}{AB} = \text{a positive quantity;}$$

$$\text{Sec. } DAB = \frac{AD}{AE} = \text{a negative quantity.}$$

47. It follows from the preceding article that trigonometrical tables for acute angles can be used for obtuse angles, as the numerical values of the ratios of the one are the same as those of the corresponding ratios of the other.

To find, therefore, the trigonometric ratio of any obtuse angle, we subtract the given angle from 180° and find the corresponding ratio of the remainder.

Where it is important—as in certain computations—to distinguish the obtuse angle from its supplementary acute angle, we retain the sign of the ratio.

Example 1. Find from the tables the trigonometric ratios of an angle of 100°.

2. Required from the tables the trigonometric ratios of an angle of 120° 13′ 45″.

3. Required the trigonometric ratios of an angle of 135°.

4. Required the trigonometric ratios of an angle of 150°.

5. Required the trigonometric ratios of an angle of 120°.

6. If the cosine of an angle is — .813467, what is the angle? *Ans.* 144° 26′ 10″.

Neglecting the sign, required the obtuse angle whose

7. Cotan. is 2.31821. *Ans.* 156° 39′ 58″.4.

8. Sine is .634167. *Ans.* 140° 38′ 30″.3.

9. Tangent is 1.81511. *Ans.* 118° 51′ 7″.4.

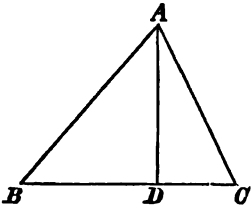
# CHAPTER VI.

## OBLIQUE-ANGLED TRIANGLES.

**Def.** An *oblique-angled triangle* is one which does not contain a right angle. It is, therefore, either an acute-angled or an obtuse-angled triangle.

Art. 48. *In any triangle, the sine of any angle is to the sine of a second angle as the side opposite the first angle is to the side opposite the second angle.*

This principle is sometimes stated in another form, thus: "The sines of the angles of a triangle are proportional to the opposite sides."



In the given triangle, $ABC$, it is required to prove

$$\frac{\text{Sin. } B}{\text{Sin. } C} = \frac{AC}{AB}.$$

Suppose that $ABC$ be a triangle, in which a perpendicular from one of the angles upon the opposite side falls within the triangle, as the perpendicular, $AD$, from $A$ upon $BC$.

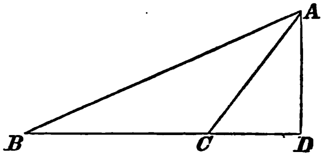
$$\text{Sin. } B = \frac{AD}{AB} \text{ ((1) Art. 4); also, sin. } C = \frac{AD}{AC};$$

$$\text{therefore } \frac{\text{sin. } B}{\text{sin. } C} = \frac{\frac{AD}{AB}}{\frac{AD}{AC}} = \frac{AC}{AB}.$$

By drawing a perpendicular from $C$ upon $A\,B$, it can also be proved, in the same way, that

$$\frac{\text{Sin. } A}{\text{Sin. } B} = \frac{B\,C}{A\,C}.$$

Next, let the triangle $A\,B\,C$ be a triangle in which the perpendicular falls without the triangle, on the side produced.



In the figure, $A\,D$, the perpendicular, falls without the triangle. The angle, $A\,C\,B$, is therefore an obtuse angle.

$$\text{Now, sin. } A\,B\,C = \frac{A\,D}{A\,B};$$

$$\text{also, sin. } B\,C\,A = \frac{A\,D}{A\,C} \text{ ((1) Art. 44);}$$

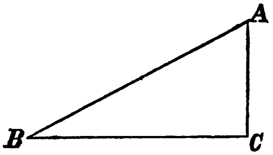
$$\text{therefore } \frac{\text{sin. } A\,B\,C}{\text{sin. } A\,C\,B} = \frac{\frac{A\,D}{A\,B}}{\frac{A\,D}{A\,C}} = \frac{A\,C}{A\,B}.$$

By drawing a perpendicular from $B$, upon $A\,C$ produced, it may be proved in the same way that

$$\frac{\text{Sin. } B\,A\,C}{\text{Sin. } B\,C\,A} = \frac{B\,C}{A\,B}.$$

Lastly, let $A\,B\,C$ be a right-angled triangle; then also

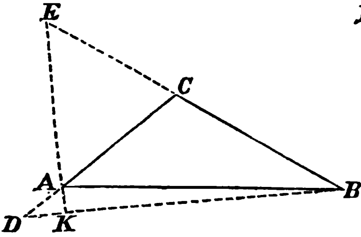
$$\frac{\text{Sin. } B}{\text{Sin. } C} = \frac{A\,C}{A\,B}.$$

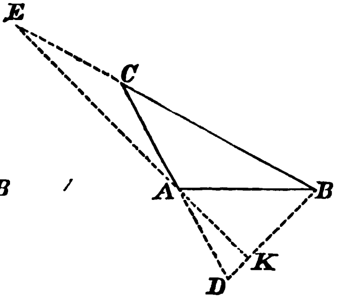


For, since $C$ is a right angle, its sine is 1 (Art. 35). Also

$$\text{sin. } B = \frac{A\,C}{A\,B}\text{; i. e., } \frac{\text{sin. } B}{1} = \frac{A\,C}{A\,B}, \text{ or } \frac{\text{sin. } B}{\text{sin. } C} = \frac{A\,C}{A\,B}.$$

49. *In any triangle, the* SUM *of any* TWO SIDES *is to their* DIFFERENCE *as the* TANGENT *of* HALF the sum of the OPPOSITE ANGLES is *to the* TANGENT *of* HALF *their* DIFFERENCE.





Let $ACB$ be any triangle. Then

$$\frac{BC+CA}{BC-CA}=\frac{\text{tan. } \frac{1}{2}(A+B)}{\text{tan. } \frac{1}{2}(A-B)}.$$

Produce $CA$ to $D$, making $CD$ equal to $CB$. Produce $BC$ to $E$, making $CE$ equal to $CA$. Join $D$ and $B$, by the straight line $DB$. Join $E$ and $A$, by the straight line $EA$, and produce $EA$ to meet $DB$ at $K$.

By the construction, $BE=BC+CA$
and $AD=BC-CA$.

Now the sum of the angles $CAB$ and $CBA$ equals the sum of the angles $D$ and $CBD$, because each sum is the supplement of the angle $ACB$ (Euc. 32, I. Ch. 18, I.).

But $CDB+CBD=2\,D$ (Euc. 5, I. Ch. 25, I.).

Therefore $D=\frac{1}{2}(CAB+CBA)=\frac{1}{2}(A+B)$.

Also $\frac{1}{2}(A+B)+\frac{1}{2}(A-B)=A$;

That is, $D + \frac{1}{2}(A - B) = CAB = D + ABD$ (Euc. 32, I. Ch. 18, I. Cor. 1);

therefore, $ABD = \frac{1}{2}(A - B)$.

Again, the angle $CEA = CAE = DAK$;

also $CBK = ADK$;

therefore, $BEK + EBK = DAK + ADK$; consequently $EKB = EKD$, and each is a right angle, according to the definition of a right angle.

Also the triangles $KBE$ and $ADK$ are similar (Euc. 4, I. Ch. 4, III.).

(1) Now $\frac{AK}{DK} = \text{tan. } D = \text{tan. } \frac{1}{2}(A + B)$;

(2) also $\frac{AK}{KB} = \text{tan. } ABK = \text{tan. } \frac{1}{2}(A - B$;

therefore, dividing (1) by (2), $\frac{KB}{DK} = \frac{\text{tan. } \frac{1}{2}(A+B)}{\text{tan. } \frac{1}{2}(A-B)}$.

But, from the similarity of the triangles, $KBE$ and $ADK$,

$$\frac{KB}{KD} = \frac{BE}{AD} = \frac{BC + CA}{BC - CA};$$

therefore $\frac{BC + CA}{BC - CA} = \frac{\text{tan. } \frac{1}{2}(A + B)}{\text{tan. } \frac{1}{2}(A - B)}$.

## SOLUTION OF OBLIQUE-ANGLED TRIANGLES.

50. To solve an oblique-angled triangle, it is necessary to know *three* parts, of which *one* at least must be a side.

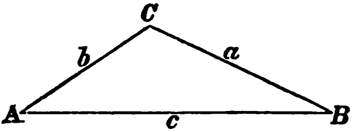
The *three angles* of a triangle determine only its *form*. The same three angles may belong to a number of triangles, of different sides and of different areas, but all similar.

We shall then have different cases of the solution of oblique-angled triangles, which may be classified as follows:

1. When *two angles* and a *side* are given;
2. When *two sides* and an *angle* are given;
3. When the *three sides* are given.

51. *Two angles and a side of an oblique-angled triangle being known, to solve the triangle.*

The third angle is found by subtracting the sum of the given angles from 180°.



The sides are found by the theorem of Art. 48. Thus in the triangle $ACB$, suppose the side $a$ is given and any two angles as $B$ and $A$.

Then $C = 180° - (A + B)$.

Also, by Art. 48, $\frac{\sin. A}{\sin. B} = \frac{a}{b}$; $\therefore b = \frac{a \sin. B}{\sin. A}$.

Again, $\frac{\sin. C}{\sin. A} = \frac{c}{a}$; $\therefore c = \frac{a \sin. C}{\sin. A}$.

**Solve the triangle when there are given:**

**Example 1.** A side = 121, and the adjacent angles = 15°, and 55° 31′. *Ans.* Sides = 33.219 and 105.797.

2. A side = 14.3, an adjacent angle = 43° 10′ 15″, an opposite angle = 73° 13′ 4″. *Ans.* Sides = 10.219 and 13.3798.

3. A side = 31.57, angle opposite = 55°, and the other sides equal.

4. $a = 15.189$, $B = 75° 10′$, $C = 33° 14′ 7″$.

5. $c = 568.9$, $B = 3\,A$, and $C = 6\,A$.

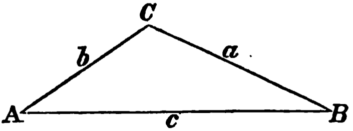
6. $b = 7.93$, $A = 3\,B$, and $B = 2\,C$.

52. *Two sides and an angle of an oblique-angled triangle being known, to solve the triangle.*

Under this head we have two cases: 1. When the given angle is *included* by the given sides; 2. When the given angle is *opposite* to one of the given sides.

53. *Two sides, and the included angle, of an oblique-angled triangle being known,* to solve the triangle.

We then have the *sum* of the other angles given, or the *sum* of the *angles opposite* the *given sides* (Euc. 32, I. Ch. 18, I.). By the theorem of Art. 49 we find half the difference of these same angles. Half the difference added to half the sum of the angles will give the greater, and half the difference subtracted from half the sum will give the less angle. The remaining side may then be found by the theorem of Art. 48.



Thus in triangle $A\,B\,C$, if we have the sides, $a$ and $b$, and the included angle $C$ given, then by Art. 49,

$$\frac{a+b}{a-b} = \frac{\tan.\ \frac{1}{2}(A+B)}{\tan.\ \frac{1}{2}(A-B)} = \frac{\tan.\ \frac{1}{2}(180^\circ - C)}{\tan.\ \frac{1}{2}(A-B)}$$

$$= \frac{\cot.\ \frac{1}{2}C}{\tan.\ \frac{1}{2}(A-B)};$$

$$\text{therefore } \tan.\ \tfrac{1}{2}(A-B) = \frac{a-b}{a+b}\tan\ \tfrac{1}{2}(A+B)$$

$$= \frac{a-b}{a+b}\cot.\ \tfrac{1}{2}C.$$

$\frac{1}{2}(A-B)$ added to $\frac{1}{2}(A+B)$ or to $\frac{1}{2}(180^\circ - C)$ will equal $A$, and $\frac{1}{2}(A-B)$ subtracted from $\frac{1}{2}(A-B)$ or from $\frac{1}{2}(180^\circ - C)$ will equal $B$.

We shall then have the two sides, $a$ and $b$, and all the angles given to find the side $c$.

Then by the theorem of Art. 48,

$$\frac{\sin. A}{\sin. C}=\frac{a}{c};\text{ or }c=\frac{a\sin. C}{\sin. A}.$$

**Solve the triangle when there are given:**

**EXAMPLE 1.** $c = 13$, $b = 7$, and angle $A = 52°$.

*Ans.* $a = 10.293$, $C = 95° \; 35' \; 43''$, $B = 32° \; 24' \; 17''$.

2. $c = 12$, $b = 8$, and angle $A = 42°$.

*Ans.* $a = 8.0818$, $B = 41° \; 28' \; 47''$, $C = 96° \; 31' \; 12''$.

3. $b = 4$, $c = 16$. $A = 55°$.

*Ans.* $B = 13° \; 26' \; 43''$, $C = 111° \; 33' \; 17''$, $a = 14.092$.

4. $b = 9$, $c = 19$, $A = 50°$.

*Ans.* $a = 14.905$, $B = 27° \; 33' \; 6''$, $C = 102° \; 26' \; 54''$.

5. $a = 11$, $c = 17$, $B = 70''$.

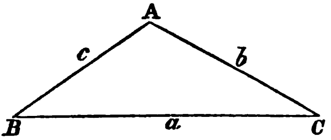
*Ans.* $b = 16.795$, $A = 37° \; 59' \; 3''$, $C = 72° \; 0' \; 57$.

6. $a = 10$, $b = 18$, $C = 30°$.

7. $b = 543$, $c = 721$, $A = 65°$.

8. $b = 543$, $c = 721$, $A = 75°$.

54. *Two sides, and an angle opposite one of the given sides, of an oblique-angled triangle being known, to solve the triangle.*



The triangle may be solved by the theorem of Art. 48.

Thus in the triangle $A \, B \, C$, suppose we have given the two sides $a$ and $c$, and the angle $A$; then, by Art. 48,

$$\frac{c}{a}=\frac{\sin. C}{\sin. A};\text{ therefore }\sin. C=\frac{c}{a}\sin A.$$

Having $C$ and $A$, we then find $B = 180° - (A + C)$.

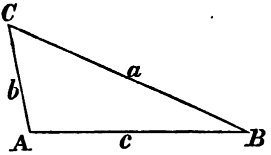
Then (Art. 48), $\frac{b}{a}=\frac{\sin. B}{\sin. A}$; or $b=\frac{a\sin. B}{\sin. A}$.

55. Of the parts mentioned in Art. 54, *when the side opposite the given angle is less than the other given side, and greater than the product of that side into the sine of the given angle, two triangles can be constructed from the given parts, and therefore two solutions will be possible, both of which will be correct.*

In other cases there will be but *one* solution.

The first case is sometimes spoken of as the *ambiguous* case.

We will consider the cases which are not ambiguous, and then the ambiguous case.



1. Suppose, in the triangle $ABC$, there are given the two sides, $c$ and $a$, and the angle $A$, and that $c$ is *less* than $a$.

By Art. 48,

$$\frac{\sin. C}{\sin. A} = \frac{c}{a}; \text{ therefore } \sin. C = \frac{c}{a} \sin. A. \quad (1)$$

But if $c$ is less than $a$, $\frac{c}{a}$ is less than 1; and, therefore, $\frac{c}{a}$ sin. $A$ is less than sin. $A$, or, since sin. $C$ is equal to $\frac{c}{a}$ sin. $A$, sin. $C$ is less than sin. $A$.

Therefore $C$ is less than $A$ (Art. 8), and must be an *acute* angle. For if $C$ were an obtuse angle, as its sine is less than sin. $A$, $C$ would be greater than the supplement of $A$ (Art. 45). Consequently $C$ and $A$, two angles of a triangle, would be together greater than two right angles, which is impossible (Euc. 17, I. Ch. 18, I.). Therefore, in this case, there is no ambiguity.

Or, by geometry, $C$ is less than $A$ (Euc. 18, I. Ch. 26, I.).

2. Again, suppose $c = a$. Then, as $\frac{c}{a} = 1$, from (1) sin. $C =$ sin. $A$ and $C = A$. $C$ could not be the supplement of $A$, for then the two angles of a triangle would be equal to two right angles, which is impossible.

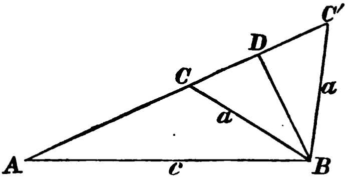
By geometry $C = A$ (Euc. 5, I. Ch. 25, I.).

3. Lastly, suppose $a = c$ sin. $A$.

Substituting this value in (1) sin. $C = 1$; therefore, $C$ is a right angle (Art. 35). And in this case there is no ambiguity.

4. Now, suppose $a$ is less than $c$, but greater than $c$ sin. $A$.

If $c$ and $a$ and the angle $A$ are given, and $a$ is greater than $c$ sin. $A$, then $a$ is greater than $BD$ (*see* figure), the perpendicular from $B$ upon $AD$, for this perpendicular is equal to $c$ sin. $A$ (Art. 30).



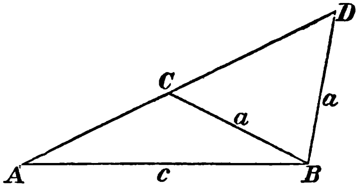
From the equation sin. $C = \frac{c}{a}$ sin. $A$, as $\frac{c}{a}$ is greater than 1, $\frac{c}{a}$ sin. $A$, or its equal, sin. $C$, is greater than sin. $A$.

Now, $A$ cannot be an obtuse angle when $a$ is less than $c$, for then would $C$ be also an obtuse angle (Euc. 18, I. Ch. 26, I.), that is, two angles of a triangle would be together greater than two right angles, which is impossible. Therefore, $A$ is an acute angle. Also, as sin. $C$ has been shown to be greater than sin. $A$, $C$ is an angle greater than $A$, but may be either an acute angle, or an obtuse angle the supplement of the acute angle, as the sine of an angle and the sine of its supplement are the same in value and in sign (Art. 46).

Thus, in the figure, if $a$ is less than $c$ and greater than $c$ sin. $A$, we shall have two triangles, $A\,C'\,B$ and $A\,C\,B$, having the angle $A$ and the side $c$ in common, and the side $B\,C'$ equal to $B\,C$, but the angle $B\,C'\,A$ the supplement of $B\,C\,A$.

5. When $a$ is less than $c$ sin. $A$, there is no triangle formed.

56. *To solve the triangle,* when the parts are given as in the preceding article; that is, *when the side opposite the given angle is less than the other given side, and greater than the product of the sine of the given angle by that side.*



Suppose we have given the sides $c$ and $a$ and the angle $A$, and that $a$ is less than $c$, but greater than $c$ sin. $A$.

From $A$ draw the line $A\,D$, making the given angle with $A\,B\,(c)$, and produce it indefinitely. From $B$ as a centre with a radius equal to $a$, describe a circle cutting $A\,D$ at $C$ and at $D$. Join $B$ and $C$, and $B$ and $D$, by the straight lines $B\,C$ and $B\,D$ respectively. Then we have two triangles, each of which contains the given parts.

Since $B\,C$ equals $B\,D$, the angle $B\,C\,D$ equals $B\,D\,C$; therefore $A\,C\,B$, which is the supplement of $B\,C\,D$, is also the supplement of $B\,D\,C$.

(1) $\frac{\text{Sin. } D}{\text{Sin. } A} = \frac{c}{a}$; (Art. 48) $\therefore$ sin. $D = \frac{c}{a}$ sin. $A$.

We thus find $D$. $A\,B\,D = 180° - (A + D)$.

(2) Then $\frac{A\,D}{c} = \frac{\text{sin. } A\,B\,D}{\text{sin. } D}$;

$$\therefore\ A\,D = \frac{c \text{ sin.} A\,B\,D}{\text{sin. D}}.$$

Thus the triangle $A\,B\,D$ is solved.

To solve the triangle $A\,B\,C$, we shall have, as in (1) for sin. $D$, sin. $C = \frac{c}{a}$ sin. $A$; but as $A\,C\,B$ is the supplement of $A\,D\,B$, after finding $D$, we subtract $D$ from 180° to find $A\,C\,B$. Then $A\,B\,C = 180° - (A\,C\,B + A)$, or $= D\,C\,B - A$.

Lastly, $\frac{A\,C}{B\,C} = \frac{\text{sin. } A\,B\,C}{\text{sin. } A}$;

$$\therefore\ A\,C = \frac{a \text{ sin.} A\,B\,C}{\text{sin.} A}.$$

Solve an oblique-angled triangle when there are given:

EXAMPLE 1. $A = 74° 45'$, $a = 475$, $b = 432$.

*Ans.* $B = 61° 20' 10''$, $C = 43° 54' 50''$, $c = 341.47$.

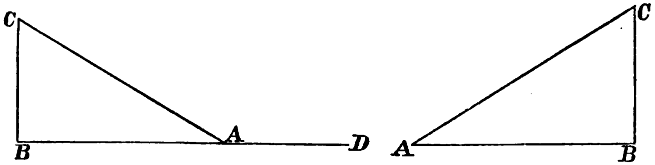
2. $B = 67° 30'$, $b = 310$, $c = 292$.
3. $C = 41° 15' 5''$, $c = 891$, $a = 311$.
4. $B = 49° 30'$, $b = 5$, $c = 6$.

*Ans.* Two solutions, $A = 64° 38' 55''$, $C = 65° 51' 5''$, $a = 5.9422$.
or $16° 21' 5''$, or $114° 8' 55''$, or $1.8512$.

5. $B = 35^\circ$, $b = 111$, $c = 123$.
*Ans.* Two solutions, $A = 105^\circ\, 32'\, 12''$, $C = 39^\circ\, 27'\, 48''$, $a = 186.45$.
or $4^\circ\, 27'\, 48''$, or $140^\circ\, 32'\, 12''$, or $15.06$.
6. $C = 25^\circ$, $c = 115$, $b = 191$.
*Ans.* Two solutions, $A = 110^\circ\, 25'\, 10''$, $B = 44^\circ\, 34'\, 50''$, $a = 255.015$.
or $19^\circ\, 34'\, 50''$, or $135^\circ\, 25'\, 10''$, or $91.194$.
7. $C = 31^\circ\, 30'$, $c = 115$, $b = 191$.
8. $A = 40^\circ$, $a = 129$, $c = 165$.

57. In order to solve an oblique-angled triangle when its three sides are given, we establish three principles: 1. That *the tangent of an angle is equal to the sine of the angle divided by its cosine;* 2. That *the sine of half an angle is equal to the square root of half the difference between 1 and the cosine of the angle;* and, 3. That *the cosine of half an angle is equal to the square root of half the sum of 1 and the cosine of the angle; sum and difference* being used in the *algebraic* sense.

58. *The tangent of an angle is equal to its sine divided by its cosine.*



Suppose $A$ to be a given angle, acute in the right-hand figure, obtuse in the left-hand figure.

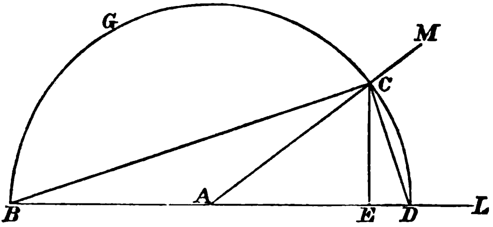
$$\text{Then tan. } A = \frac{\text{sin. } A}{\text{cos. } A}.$$

Make the triangle $A\,B\,C$, right-angled at $B$, the triangle of reference.

$$\text{Tan. } A = \frac{CB}{AB} \text{ ((2) Art. 4 and (2) Art. 45)} = \frac{\frac{CB}{AC}}{\frac{AB}{AC}}$$

$$= \frac{\sin. A}{\cos. A} \text{ ((1) and (4) Articles 4 and 44).}$$

59. *The sine of half an angle is equal to the square root of half the difference between 1 and the cosine of the whole angle.*



1. Let $MAL$ be an acute angle; then

$$\sin. \tfrac{1}{2} A = \sqrt{\frac{1 - \cos. A}{2}}.$$

With $A$ as centre and any radius $AC$, describe a semicircle, $BGD$, meeting $AL$ in $D$, and $AL$ produced in $B$, and cutting $AM$ in $C$. Draw the straight lines $BC$ and $CD$, and from $C$ draw $CE$ perpendicular to $AD$, meeting $AD$ at $E$.

The angle $B = \frac{1}{2} A$ (Euc. 5 and 32, I. Ch. 25, and Cor. I. 18, I.);

$$\sin. \tfrac{1}{2} A = \sin. B = \frac{CE}{CB};$$

$$(\sin. B)^2 \text{ (written } \sin.^2 B) = \frac{CE^2}{CB^2} = \frac{BE \times ED}{BE \times BD}$$

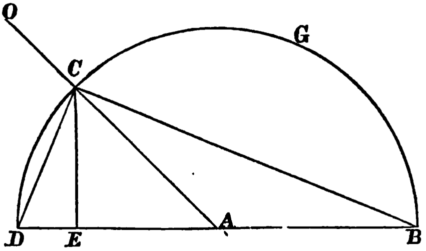
$$\text{(Euc. 8, VI. Ch. 13, III.)} = \frac{ED}{BD};$$

therefore $\sin.^2 B = \frac{ED}{BD} = \frac{AD - AE}{2AD} = \frac{1}{2}\left(1 - \frac{AE}{AC}\right)$
$= \frac{1}{2}(1 - \cos. A)$;

therefore $\sin. \frac{1}{2} A = \sin. B = \sqrt{\frac{1 - \cos. A}{2}}$.

2. Let us take the obtuse angle $BAO$; then, also,

$$\sin. \tfrac{1}{2} A = \sqrt{\frac{1 - \cos. A}{2}}.$$



From $A$ as centre, with radius $AC$, describe the semicircle $BGD$, and complete the figure, as in the previous article.

The angle $BDC = \frac{1}{2} BAC$;

therefore $\sin. \frac{1}{2} A = \sin. BDC = \frac{CE}{CD}$.

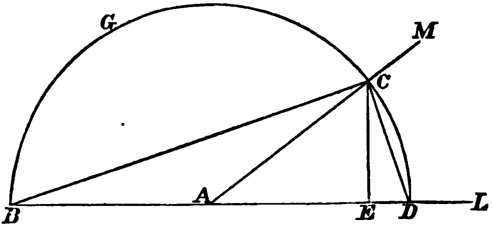
$$\text{Sin.}^2 BDC = \frac{CE^2}{CD^2} = \frac{DE \times BE}{DE \times BD} = \frac{BE}{BD}$$
$$= \frac{BA + AE}{2BA} = \tfrac{1}{2}\left(1 + \frac{AE}{AC}\right).$$

Now $AE$ is a minus quantity (Art. 43), and $\frac{AE}{AC} =$ $-\frac{-AE}{AC} = -\cos. A$ ((4) Art. 44).

Therefore $\sin. \frac{1}{2} A = \sin. BDC = \sqrt{\frac{1 - \cos. A}{2}}$.

60. *The cosine of half an angle is equal to the square root of one-half the sum of 1 and the cosine of the whole angle.*

First, let the angle be an acute angle, as the angle $MAL$.



Then $\cos. \frac{1}{2} A = \sqrt{\frac{1 + \cos. A}{2}}$.

Construct the figure as for the first part of Art. 59.

$\text{Cos.} \frac{1}{2} A = \cos. B = \frac{BE}{BC}$.

$$\text{Cos.}^2 B = \frac{BE^2}{BC^2} = \frac{BE^2}{BE \times BD} = \frac{BE}{BD} = \frac{BA + AE}{2AC}$$

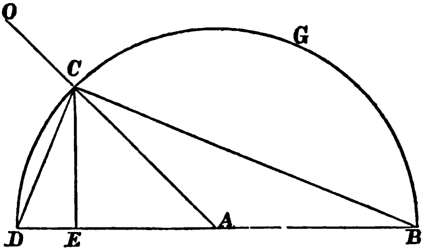
$$= \tfrac{1}{2}\left(1 + \frac{AE}{AC}\right).$$

Now $\frac{AE}{AC} = \cos. A$ ((4) Art. 4);

therefore $\cos. \frac{1}{2} A = \cos. B = \sqrt{\frac{1 + \cos. A}{2}}$.

Next let $A$ be an obtuse angle, as the angle $BAO$.

Then also $\cos. \frac{1}{2} A = \sqrt{\frac{1 \times \cos. A}{2}}$.



Construct the figure as for the second part of Art. 59.

$$\text{Cos. } \tfrac{1}{2} A = \text{cos. } B D C = \frac{D E}{C D}.$$

$$\text{Cos. }^2 B D C = \frac{D E^2}{C D^2} = \frac{D E^2}{D E \times B D} = \frac{D E}{B D}$$

$$= \frac{A D - A E}{2 A D} = \tfrac{1}{2}\left(1 - \frac{A E}{A C}\right).$$

But cos. $A = \frac{A E}{A C}$, and is itself a minus quantity (Art. 43, (4) Art. 45), and therefore $- \frac{A E}{A C}$ is $+$ cos. $A$.

Therefore cos. $\tfrac{1}{2} A = \text{cos. } B D C = \sqrt{\frac{1 + \text{cos. } A}{2}}$.

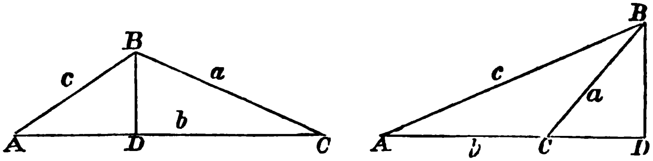
61. *The three sides of a triangle being known, to solve the triangle.*

Let $A B C$ be an oblique-angled triangle, of which the sides $a$, $b$, and $c$ are given. It is required to find the angles.

From $B$ draw a perpendicular $B D$ to $A C$, or $A C$ produced.

(1) $A D = A B$ cos. $A = c$ cos. $A$ (Art. 30).

(2) Also, $a^2 + 2 b \times A D = b^2 + c^2$ (Euc. 13, II. Ch. 15, III.).



In (2) substitute the value of $AD$ already found in (1) and $a^2 + 2\,b\,c$, cos. $A = b^2 + c^2$.

(3) Therefore $\cos. A = \frac{b^2 + c^2 - a^2}{2\,b\,c}$.

(4) Subtracting each member of (3) from (1), and we have $1 - \cos. A = \frac{2\,b\,c - b^2 - c^2 + a^2}{2\,b\,c} = \frac{a^2 - (b-c)^2}{2\,b\,c}$

$$= \frac{(a+c-b)\,(a+b-c)}{2\,b\,c}.$$

Now let $\frac{a+b+c}{2} = s$;

then $\frac{a+c-b}{2} = s - b$; $\frac{a+b-c}{2} = s - c$; and $\frac{b+c-a}{2} = s - a$.

Dividing (4) by 2, extracting the square root of both members, and substituting the values given for $\frac{a+c-b}{2}$, and $\frac{a+b-c}{2}$,

$$\sqrt{\frac{1 - \cos. A}{2}} = \sqrt{\frac{(s-b)\,(s-c)}{b\,c}}.$$

But $\sqrt{\frac{1 - \cos. A}{2}} = \sin. \tfrac{1}{2} A$ (Art. 59).

(5) Therefore $\sin.\ \tfrac{1}{2} A = \sqrt{\dfrac{(s-b)\ (s-c)}{b\ c}}.$

Again, adding 1 to both members of (3),

$$(6)\ 1 + \cos. A = \frac{2bc + b^2 + c^2 - a^2}{2bc} = \frac{(b+c)^2 - a^2}{2\,b\,c}$$
$$= \frac{(a+b+c)\,(b+c-a)}{2bc}.$$

Dividing (6) by 2, extracting the square root, and substituting values for $\dfrac{a+b+c}{2}$ and $\dfrac{b+c-a}{2}$,

$$\sqrt{\frac{1+\cos. A}{2}} = \sqrt{\frac{s\ (s-a)}{b\ c}}.$$

But $\sqrt{\dfrac{1+\cos. A}{2}} = \cos.\ \tfrac{1}{2} A$ (Art. 60).

(7) Therefore $\cos. \tfrac{1}{2} A = \sqrt{\dfrac{s\,(s-a)}{b\ c}}.$

Dividing (5) by (7) we have

$$(8)\ \frac{\text{Sin.}\ \frac{1}{2} A}{\text{Cos.}\ \frac{1}{2} A} = \tan.\ \tfrac{1}{2} A = \sqrt{\frac{(s-b)\ (s-c)}{s\ (s-a)}}.$$

We can prove in a similar manner:

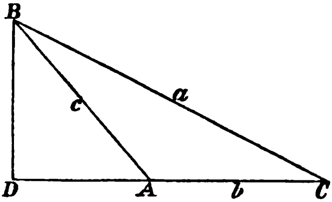
$$\sin.\ \tfrac{1}{2} B = \sqrt{\frac{(s-c)\ (s-a)}{c\,a}};\ \cos.\ \tfrac{1}{2} B = \sqrt{\frac{s\ (s-b)}{c\,a}};$$

and $\tan.\ \tfrac{1}{2} B = \sqrt{\dfrac{(s-c)\ (s-a)}{s\ (s-b)}}.$

Also:

$$\sin.\ \tfrac{1}{2} C = \sqrt{\frac{(s-a)\ (s-b)}{a\ b}};\ \cos.\ \tfrac{1}{2} C = \sqrt{\frac{s\ (s-c)}{a\ b}};$$

and $\tan.\ \tfrac{1}{2} C = \sqrt{\dfrac{(s-a)\ (s-b)}{s\ (s-c)}}.$



62. If the angle, as $A$, in the figure, be an obtuse angle, the expressions in (5), (7), and (8), of the preceding article will still be of the same form.

For we shall have:

(1) $a^2 - 2b \times AD = b^2 + c^2$ (Euc. 12, II. Ch. 16, III.).

Now $AD = c \times \cos. BAD = c \times - \cos. A$ (Art. 46), and substituting this value in (1) we shall have $\cos. A = \frac{b^2 + c^2 - a^2}{2bc}$, the same expression as in equation (3) of the preceding article. Therefore we shall have the same expressions for sin. $\frac{1}{2} A$, cos. $\frac{1}{2} A$, tan. $\frac{1}{2} A$, as in the preceding article.

Find the angles of a triangle when—

**Example** 1. $a = 6, b = 5, c = 4$.
*Ans.* $A = 82° 49' 10''$, $C = 41° 24' 36''$.
$B = 55° 46' 16''$.

2. $a = 11, b = 13, c = 16$. *Ans.* $A = 43° 2' 56''$, $C = 83° 10' 22''$.
$B = 53° 46' 42''$.

3. $a = 25, b = 26, c = 27$. *Ans.* $A = 56° 15' 4''$, $C = 63° 53' 46''$.
$B = 59° 51' 10''$.

4. $a = 222, b = 318, c = 406$.
*Ans.* $A = 32° 57' 7''$, $C = 95° 51' 55''$.
$B = 51° 10' 58''$.

5. $a = 400, b = 340, c = 250$.

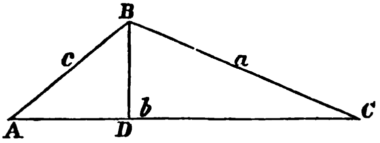
6. $a = 1410, b = 2160, c = 3142$.

7. $a = 520, b = 348, c = 192$.

8. $a = 412, b = 562, c = 330$.

63. *Second method of solving a triangle when the three sides are known.*

Let $ABC$ be any triangle of which the three sides



are known. From $B$ draw a perpendicular $BD$ upon $AC$, the longest side, dividing the triangle into two right-angled triangles, $ABD$ and $BDC$.

(1) $BC^2 = CD^2 + BD^2$ (Euc. 47, I. Ch. 14, III.).

(2) $AB^2 = AD^2 + BD^2$.

Subtracting equation (2) from equation (1),

$$BC^2 - AB^2 = CD^2 - AD^2.$$

Factoring,

$$(BC + AB)(BC - AB) = AC(CD - DA).$$

$$(3)\quad CD - DA = \frac{(BC + AB)(BC - AB)}{AC} = \frac{(a + c)(a - c)}{b}.$$

Equation (3) will give the difference of the segments of the base. *Half the base* (or half the sum of the segments) *added to half the difference of the segments of the base* will give the *greater* segment, $CD$; and *half the difference of the segments* of the base *subtracted from half the base* will give the *smaller* segment, $AD$. Then, in each of the two right-angled triangles $ABD$ and $BCD$, we have the hypotenuse and a side given, to solve the triangle (Art. 26).

Solve the triangle when the sides are given:

EXAMPLE 1. $a = 8$, $b = 6$, $c = 4$.

*Ans.* Segments of $a = 5.25$, $A = 104° 28' 39''$, $B = 46° 34' 3''$,
2.75, $C = 28° 57' 18''$.

2. $a = 219$, $b = 91$, $c = 245$.

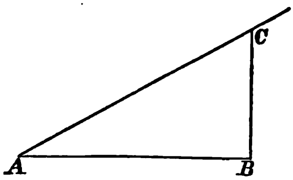
*Ans.* Segments of $c = 203.479$, $A = 62° 51' 11''$, $C = 95° 26' 45''$.
41.521, $B = 21° 42' 4''$.

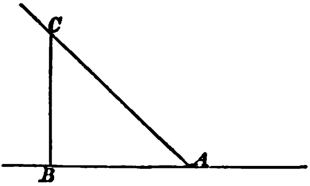
3. $a = 1140$, $b = 718.9$, $c = 627$.

# CHAPTER VII.

## RELATIONS OF THE TRIGONOMETRICAL RATIOS TO EACH OTHER.—TRIGONOMETRICAL RATIOS OF TWO ANGLES.

**Art. 64.** *The trigonometrical ratios of an angle interchangeable.*





Let $B\,A\,C$ be the triangle of reference for the angle $A$.

$$\text{Sin. } A = \frac{B\,C}{A\,C};\ \text{cos. } A = \frac{A\,B}{A\,C}.$$

Squaring and adding,

$$(a)\ \text{Sin.}^2 A + \text{cos.}^2 A = \frac{B\,C^2 + A\,B^2}{A\,C^2} = 1;$$

$$\text{sin. } A = \pm\sqrt{1 - \text{cos.}^2 A};$$

$$\text{cos. } A = \pm\sqrt{1 - \text{sin.}^2 A}.$$

$$(b)\ \text{Tan. } A = \frac{\text{sin. } A}{\text{cos. } A}\ \text{(Art. 58)}.$$

$$(c)\ \text{Sec. } A = \frac{A\,C}{A\,B} = \frac{1}{\dfrac{A\,B}{A\,C}} = \frac{1}{\text{cos. } A}.$$

(*d*) $\text{Cot. } A = \frac{AB}{BC} = \frac{\frac{AB}{AC}}{\frac{BC}{AC}} = \frac{\text{cos. } A}{\text{sin. } A}.$

(*e*) $\text{Cosec. } A = \frac{AC}{BC} = \frac{1}{\frac{BC}{AC}} = \frac{1}{\text{sin. } A}.$

(*f*) $\text{Tan. } A = \frac{BC}{AB} = \frac{1}{\frac{AB}{BC}} = \frac{1}{\text{cot. } A}.$

(*f*) also follows from (*b*) and (*d*).

(*g*) $\text{Sec. } A = \frac{AC}{AB} = \sqrt{\frac{AC^2}{AB^2}} = \sqrt{\frac{AB^2 + BC^2}{AB^2}}$

$= \sqrt{1 + \text{tan.}^2 A}.$

(*h*) $\text{Cosec. } A = \frac{AC}{BC} = \sqrt{\frac{AC^2}{BC^2}} = \sqrt{\frac{BC^2 + AB^2}{BC^2}}$

$= \sqrt{1 + \text{cot.}^2 A}.$

65. By means of the equations established in the preceding article, it is possible to express all the trigonometrical ratios in terms of any one ratio.

Thus to express the trigonometrical ratios in terms of the tangent:

$$\text{Sin. } A = \frac{1}{\text{cosec. } A} \text{ ((e) Art. 64)} = \frac{1}{\sqrt{1 + \text{cot.}^2 A}}$$

$$= \frac{1}{\sqrt{1 + \frac{1}{\text{tan.}^2 A}}} = \frac{\text{tan. } A}{\sqrt{1 + \text{tan.}^2 A}}.$$

$$\text{Cos. } A = \frac{1}{\text{sec. } A} = \frac{1}{\sqrt{1 + \text{tan.}^2 A}}.$$

$$\text{Sec. } A = \sqrt{1 + \text{tan.}^2 A}.$$

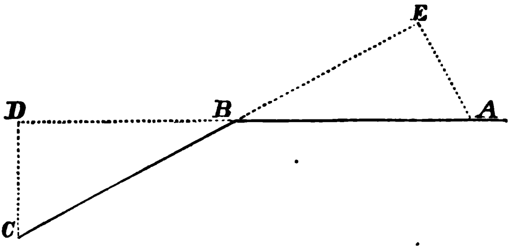
$$\text{Cosec. } A = \sqrt{1 + \text{cot.}^2 A} = \sqrt{1 + \frac{1}{\text{tan.}^2 A}}$$

$$= \sqrt{\frac{1 + \text{tan.}^2 A}{\text{tan. } A}}.$$

$$\text{Cot. } A = \frac{1}{\text{tan. } A}.$$

1. Given sin. $A = \frac{1}{3}$; find the remaining trigonometric ratios for $A$.
2. Given cos. $B = \frac{2}{3}$; find the remaining trigonometric ratios for $B$.
3. Given tan. $C = \frac{3}{2}$; find the remaining trigonometric ratios for $C$.
4. Given sec. $D = 3$; find the remaining trigonometric ratios for $D$.
5. Given cot. $E = \frac{7}{5}$; find the remaining trigonometric ratios for $E$.
6. Given cosec. $F = \frac{7}{4}$; find the remaining trigonometric ratios for $F$.
7. Given cos. $G = -\frac{3}{5}$; find the remaining trigonometric ratios for $G$.
8. Given tan. $H = \frac{10}{-8}$; find the remaining trigonometric ratios for $H$.

66. The definitions of the trigonometrical ratios, as applied to an acute angle and to an obtuse angle, may be extended to any angle whatever.



Thus, if $A\ B\ C$ (the salient angle) be an angle *great-*

*er than two right angles*, we form the triangle of reference, $BCD$, by drawing a perpendicular, from any point $C$ in one side of the angle, to the other side, $AB$ produced (Art. 3). Then—

(1) Sin. $ABC = \frac{CD}{BC} =$ a negative quantity (Art. 41).

(2) Tan. $ABC = \frac{CD}{BD} =$ a positive quantity.

(3) Sec. $ABC = \frac{CB}{BD} =$ a negative quantity.

(4) Cos. $ABC = \frac{BD}{BC} =$ a negative quantity.

(5) Cot. $ABC = \frac{BD}{CD} =$ a positive quantity.

(6) Cosec. $ABC = \frac{BC}{CD} =$ a negative quantity.

Also, if we draw a perpendicular from $A$ upon $CB$ produced, the triangle $ABE$ will be the triangle of reference, and will be similar to $CBD$. (Euc. 4, VI. Ch. 4, III. Cor.)

Consequently—

(7) Sin. $ABC = \frac{AE}{AB} = \frac{CD}{BC}$.

(8) Cos. $ABC = \frac{BE}{AB} = \frac{BD}{BC}$, etc.

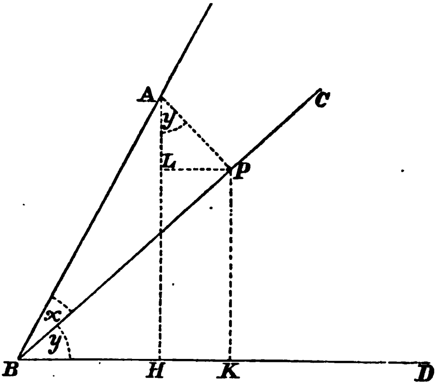
67. *To express the sine and the cosine of the sum of two angles, in terms of the sines and cosines of the given angles.*

Denote one of the angles by $x$, and the other by $y$, then we shall prove—

(*a*) Sin $(x+y) =$ sin. $x$ cos. $y +$ sin. $y$ cos. $x$.

(*b*) Cos. $(x+y) =$ cos. $x$ cos. $y -$ sin. $x$ sin. $y$.

First, let both angles be *acute*. Let $ABC$ and



$CBD$ be two acute angles, whose sum is $ABD$. It is required to find the sine of $ABD$ in terms of the sines and cosines of $ABC$, and of $CBD$.

Denote the angle $ABC$ by $x$, and $CBD$ by $y$.

In $AB$, take any point $A$, and from $A$ draw $AP$ perpendicular to $BC$, and $AH$ perpendicular to $BD$. From $P$, the foot of the perpendicular from $A$ upon $BC$, draw $PK$ perpendicular to $BD$, and $PL$ perpendicular to $AH$.

The angle $LAP$ equals $CBD$, that is, $y$.

$$\text{Sin.}\,ABD = \frac{AH}{AB} = \frac{AL+PK}{AB} = \frac{AL}{AB} + \frac{PK}{AB}$$

$$= \frac{AP}{AB} \times \frac{AL}{AP} + \frac{PK}{BP} \times \frac{BP}{AB}.$$

Now, sin. $ABD =$ sin $(x+y)$;

Also, $\frac{AP}{AB} = \sin. x$; $\frac{AL}{AP} = \cos. y$; $\frac{PK}{BP} = \sin. y$; and $\frac{BP}{AB} = \cos. x$;

Therefore, $\sin. (x + y) = \sin. x \cos. y + \sin y \cos. x$.

Again, $\cos. (x + y) = \cos. x \cos. y - \sin x \sin y$.

For, the same construction being made—

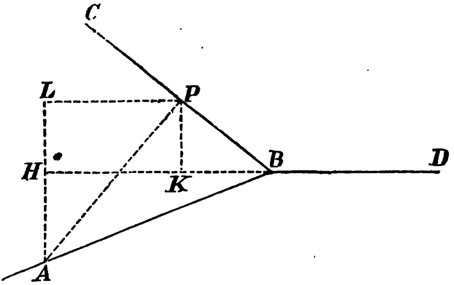
$$\text{Cos. } ABD = \frac{BH}{AB} = \frac{BK - LP}{AB} = \frac{BK}{AB} - \frac{LP}{AB}$$

$$= \frac{BP}{AB} \times \frac{BK}{BP} - \frac{AP}{AB} \times \frac{LP}{AP}.$$

But $\cos. ABD = \cos. (x + y)$;

Also, $\frac{BP}{AB} = \cos. x$; $\frac{BK}{BP} = \cos. y$; $\frac{AP}{AB} = \sin. x$; and $\frac{LP}{AP} = \sin. y$;

Therefore $\cos. (x + y) = \cos. x \cos. y - \sin. x \sin. y$.



Next let one of the angles, $ABC$, be an acute angle, and the other, $CBD$, be an obtuse angle. Denote $ABC$ by $x$, and $CBD$ by $y$.

From $A$ (any point in $AB$), draw $AP$ perpendicular to $BC$, $AH$ perpendicular to $BD$ produced. From $P$, where $AP$ meets $BC$, draw $PK$ perpendicular to $BD$ produced, and draw $PL$ perpendicular to $AH$ produced.

The salient angle, $ABD$, is the sum of $ABC$ and $CBD$, and equals $(x+y)$. $ABH$ is the triangle of reference for $ABD$ (Art. 66). Also (Euc. 4, 6. Ch. 7, III.), $LAP$, which is similar to $PKB$, the triangle of reference for $CBD$, or $y$, may be used for the triangle of reference for $y$. The angle $LAP$ is equal to the angle $PBK$.

By Art. 41, the lines $AH$, $AL$, and $BK$, $BH$, are negative. We shall put the minus sign before them to indicate the fact. Now,

$$\text{Sin. } ABD = \frac{-AH}{AB} \text{ (Art. 66)} = \frac{-AL+PK}{AB}$$

$$= \frac{-AL}{AB} + \frac{PK}{AB} = \frac{AP}{AB} \times \frac{-AL}{AP} + \frac{PK}{PB} \times \frac{PB}{AB}.$$

But $\sin.\ ABD = \sin.\ (x+y)$;

Also $\dfrac{AP}{AB} = \sin.\ x$; $\dfrac{-AL}{AP} = \dfrac{-BK}{BP}$ ((4) Art. 44)

$= \cos.\ y$; $\dfrac{PK}{PB} = \sin.\ y$; and $\dfrac{PB}{AB} = \cos.\ x$.

Therefore $\sin.\ (x+y) = \sin.\ x \cos.\ y + \sin.\ y \cos.\ x$.

$$\text{Again, cos. } ABD = \frac{-BH}{AB} = \frac{-BK-PL}{AB}$$

$$= \frac{BP}{AB} \times \frac{-BK}{BP} - \frac{AP}{AB} \times \frac{PL}{AP}.$$

But $\cos.\ ABD = \cos.\ (x+y)$;

Also, $\frac{BP}{AB} = \cos. x$; $\frac{-BK}{BP} = \cos. y$ ((4) Art. 44); $\frac{AP}{AB} = \sin. x$, and $\frac{PL}{AP} = \frac{PK}{PB} = \sin. y$;

Therefore, $\cos. (x + y) = \cos. x \cos. y - \sin. x \sin. y$.

If both $x$ and $y$ are obtuse, it may be proved in a similar manner that—

$\text{Sin.} (x + y) = \sin. x \cos. y + \sin. y \cos. x$.

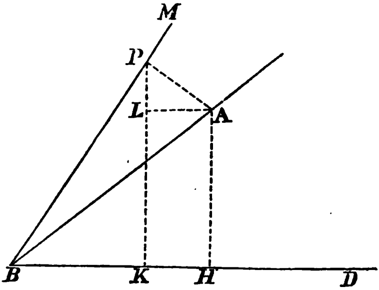
$\text{Cos.} (x + y) = \cos. x \cos. y - \sin. x \sin. y$.

68. *To express the sine and the cosine of the difference of two angles in terms of the sines and cosines of the given angles.*

Denote one of the angles by $x$, and the other by $y$; then,

($a$) $\text{Sin.} (x - y) = \sin. x \cos. y - \sin. y \cos. x$.

($b$) $\text{Cos.} (x - y) = \cos. x \cos. y + \sin. x \sin. y$.



First, let both angles be acute.

In the figure let the angle $MBD$ be denoted by $x$, and the angle $MBA$ by $y$; then $ABD$ will be $(x - y)$.

From $A$, any point in $AB$, draw $AP$ perpendicular to $BM$, and $AH$ perpendicular to $BD$. From $P$, the foot of the perpendicular $AP$, draw $PK$ perpendicular to $BD$. From $A$, draw $AL$ parallel to $BD$. $AL$ will be perpendicular to $PK$.

The angle $LPA$ is equal to $MBD$; that is, to $x$.

$$\text{Sin. } ABD = \frac{AH}{AB} = \frac{PK - PL}{AB} = \frac{PK}{AB} - \frac{PL}{AB}$$
$$= \frac{PK}{PB} \times \frac{PB}{AB} - \frac{AP}{AB} \times \frac{PL}{AP}.$$

Now, sin. $ABD = \text{sin.}\,(x - y)$;

also, $\dfrac{PK}{PB} = \text{sin.}\,x$; $\dfrac{PB}{AB} = \text{cos.}\,y$; $\dfrac{AP}{AB} = \text{sin.}\,y$;

and $\dfrac{PL}{AP} = \text{cos.}\,x$;

Therefore, $\text{sin.}\,(x - y) = \text{sin.}\,x \text{ cos.}\,y - \text{sin.}\,y \text{ cos.}\,x$.

$$\text{Cos. } ABD = \frac{BH}{AB} = \frac{BK + AL}{AB} = \frac{BK}{AB} + \frac{AL}{AB}$$
$$= \frac{BK}{BP} \times \frac{BP}{BA} + \frac{AL}{AP} \times \frac{AP}{AB}.$$

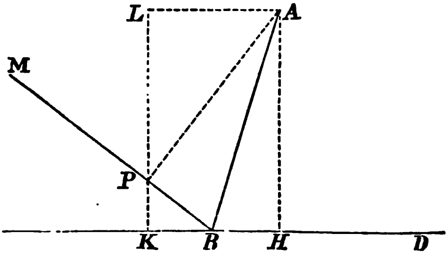
But cos. $ABD = \text{cos.}\,(x - y)$;

also, $\dfrac{BK}{BP} = \text{cos.}\,x$; $\dfrac{BP}{BA} = \text{cos.}\,y$; $\dfrac{AL}{AP} = \text{sin.}\,x$;

and $\dfrac{AP}{AB} = \text{sin.}\,y$;

Therefore, $\text{cos.}\,(x - y) = \text{cos.}\,x \text{ cos.}\,y + \text{sin.}\,x \text{ sin.}\,y$.

Next, let one angle be obtuse and the other acute.



In the figure, let $MBD$, the obtuse angle, be denoted by $x$, and let $ABM$ be the acute angle, denoted by $y$. Then $ABD$ is $(x-y)$.

The construction is similar to the construction of the preceding figure.

From $A$, any point in $AB$, draw $AH$ perpendicular to $BD$, and $AP$ perpendicular to $BM$. From $P$, the foot of the perpendicular $AP$, draw $PK$ perpendicular to $BD$ produced. Also, from $A$ draw $AL$ parallel to $BD$, and meeting $PK$ produced through $P$, at $L$. The angle $LPA$ is equal to $PBK$, and the triangle $LPA$ is similar to the triangle $PBK$, the triangle of reference for $MBD$, that is, $x$.

$$\text{Sin. } ABD = \frac{AH}{AB} = \frac{PK+PL}{AB}$$

$$= \frac{PK}{PB} \times \frac{PB}{AB} + \frac{AP}{AB} \times \frac{PL}{AP}.$$

Now, $\sin. ABD = \sin. (x-y)$;

also, $\dfrac{PK}{PB} = \sin. x$; $\dfrac{PB}{AB} = \cos. y$; and $\dfrac{AP}{AB} = \sin. y$.

Again, $-\dfrac{PL}{AP} = \dfrac{-KB}{PB} = \cos. x$ ((4) Art. 44);

therefore, $\dfrac{PL}{AP} = -\cos. x$.

Therefore, $\sin. (x-y) = \sin. x \cos. y - \sin. y \cos. x$.

$$\text{Cos. } ABD = \frac{BH}{AB} = \frac{-KB+AL}{AB}$$

$$= \frac{-KB}{PB} \times \frac{PB}{AB} + \frac{AL}{AP} \times \frac{AP}{AB}.$$

Now, $\cos. ABD = \cos. (x-y)$;

also, $\frac{-KB}{PB} = \cos. x$; $\frac{PB}{AB} = \cos. y$; $\frac{AL}{AP} = \frac{PK}{PB} = \sin. x$; and $\frac{AP}{AB} = \sin. y$.

Therefore, $\cos. (x - y) = \cos. x \cos. y + \sin. x \sin. y$.

If both $x$ and $y$ are obtuse, it may be proved, in a similar manner,

$\sin. (x - y) = \sin. x \cos. y - \sin. y \cos. x$.

$\cos. (x - y) = \cos. x \cos. y + \sin. x \sin. y$.

69. By addition of equation ($a$) of Art. 67 to equation ($a$) of Art. 68, we have

($a$) $\sin. (x + y) + \sin. (x - y) = 2 \sin. x \cos. y$.

By subtraction, we have

($b$) $\sin. (x + y) - \sin. (x - y) = 2 \sin. y \cos. x$.

By addition of equation ($b$) of Art. 67 to equation ($b$) of Art. 68, we have

($c$) $\cos. (x + y) + \cos. (x - y) = 2 \cos. x \cos. y$.

By subtraction, we have

($d$) $\cos. (x - y) - \cos. (x + y) = 2 \sin. x \sin. y$.

70. In equation of preceding article, let $x + y = a$ and $x - y = b$.

Then, finding values of $x$ and $y$ in terms of $a$ and $b$,

$$x = \frac{a + b}{2};\ y = \frac{a - b}{2}.$$

Substituting these values in the preceding equations, in order:

($a$) $\sin. a + \sin. b = 2 \sin. \frac{a + b}{2} \cos. \frac{a - b}{2}$.

($b$) $\sin. a - \sin. b = 2 \sin. \frac{a - b}{2} \cos. \frac{a + b}{2}$.

(*c*) $\cos. a + \cos. b = 2 \cos. \frac{a+b}{2} \cos. \frac{a-b}{2}.$

(*d*) $\cos. b - \cos. a = 2 \sin. \frac{a+b}{2} \sin. \frac{a-b}{2}.$

The above formulæ are useful in *logarithmic* calculations.

71. Dividing (*a*) of the preceding article by (*b*), we have

$$(a) \quad \frac{\sin. a + \sin. b}{\sin. a - \sin. b} = \tan. \frac{a+b}{2} \cot. \frac{a-b}{2}$$

$$= \frac{\tan. \frac{a+b}{2}}{\tan. \frac{a-b}{2}}.$$

Dividing (*c*) of the preceding article by (*d*), we have

$$(b) \quad \frac{\cos. a + \cos. b}{\cos. b - \cos. a} = \cot. \frac{a+b}{2} \cot. \frac{a-b}{2}$$

$$= \frac{\cot. \frac{a+b}{2}}{\tan. \frac{a-b}{2}}.$$

72. $\text{Tan.} (x + y) = \frac{\sin. (x + y)}{\cos. (x + y)}$

$$= \frac{\sin. x \cos. y + \sin. y \cos. x}{\cos. x \cos. y - \sin. x \sin. y}.$$

Divide numerator and denominator of the last fraction by $\cos. x \cos. y$:

(*a*) $\tan. (x + y) = \frac{\tan. x + \tan. y}{1 - \tan. x \tan. y}.$

In a similar manner it can be shown—

(*b*) $\tan.(x-y)=\dfrac{\tan.x-\tan.y}{1+\tan.x\tan.y}$.

Again, as $\cot.(x+y)=\dfrac{1}{\tan.(x+y)}$ ((*f*) Art. 64),

(*c*) $\cot.(x+y)=\dfrac{1-\tan.x\tan.y}{\tan.x+\tan.y}$

$$=\frac{1-\dfrac{1}{\cot.x}\times\dfrac{1}{\cot.y}}{\dfrac{1}{\cot.x}+\dfrac{1}{\cot.y}}=\frac{\cot.x\cot.y-1}{\cot.y+\cot.x}.$$

(*d*) $\cot.(x-y)=\dfrac{\cot.x\cot.y+1}{\cot.y-\cot.x}$.

(*c*) may also be proved directly, thus:

$$\cot.(x+y)=\frac{\cos.(x+y)}{\sin.(x+y)}=\frac{\cos.x\cos.y-\sin.x\sin.y}{\sin.x\cos.y+\sin.y\cos.x}.$$

Divide numerator and denominator of last fraction by $\cos.x\cos.y$.

$$\cot.(x+y)=\frac{1-\tan.x\tan.y}{\tan.x+\tan.y}.$$

Divide numerator and denominator of same fraction by $\sin.x\sin.y$.

$$\cot.(x+y)=\frac{\cot.x\cot.y-1}{\cot.y+\cot.x}.$$

(*d*) may be proved in a similar manner.

73. *Sine and cosine of an angle in terms of the sine and cosine of half the angle, and in terms of tangent of half the angle.*

In equations (*a*) and (*b*) of Art. 67 let $y = x$, then

$$(a)\ \sin. 2x = 2 \sin. x \cos. x = \frac{\dfrac{2 \sin. x}{\cos. x}}{\dfrac{1}{\cos.^2 x}} = \frac{2 \tan. x}{\sec.^2 x}$$

$$= \frac{2 \tan. x}{1 + \tan.^2 x}.$$

$$(b)\ \cos. 2x = \cos.^2 x - \sin.^2 x = 1 - 2 \sin.^2 x$$

$$= 2 \cos.^2 x - 1$$

$$= \frac{1 - \dfrac{\sin.^2 x}{\cos.^2 x}}{\dfrac{1}{\cos.^2 x}} = \frac{1 - \tan.^2 x}{1 + \tan.^2 x}.$$

74. *Tangent and cotangent of an angle in terms of the tangent and cotangent of half the angle.*

In equations (*a*) and (*c*) of Art. 72 let $y = x$, then

$$(a)\ \tan. 2x = \frac{2 \tan. x}{1 - \tan.^2 x} = \frac{\dfrac{2}{\cot. x}}{1 - \dfrac{1}{\cot.^2 x}} = \frac{2 \cot. x}{\cot.^2 x - 1}.$$

$$(b)\ \cot. 2x = \frac{1 - \tan.^2 x}{2 \tan. x} = \frac{\cot.^2 x - 1}{2 \cot. x}.$$

75. *Sine, cosine, and tangent of half an angle in terms of the cosine of the angle.*

From (*b*) of Art. 73 we have, by transposition,

$2 \sin.^2 x = 1 - \cos. 2x$; and $2 \cos.^2 x = 1 + \cos. 2x$.

$$(a)\ \text{Therefore, } \sin. x = \sqrt{\frac{1 - \cos. 2x}{2}};$$

$$(b)\ \cos. x = \sqrt{\frac{1 + \cos. 2x}{2}};$$

equations which have already been derived by geometric methods (Articles 59 and 60).

Dividing (*a*) by (*b*), we have

$$(c)\ \tan.\ x = \sqrt{\frac{1 - \cos.\ 2\,x}{1 + \cos.\ 2\,x}}.$$

**Example 1.** Find $\sin.\ 3\,x = \sin.\ (2\,x + x)$ in terms of $\sin.\ x$.

2. Find $\cos.\ 3x$ in terms of $\cos.\ x$.

3. Find $\cos.\ 4x$ in terms of $\sin.\ x$.

4. Find $\cos.\ 6x$ in terms of $\sin.\ x$.

5. Find $\sin.\ 6x$ in terms of $\sin.\ x$ and $\cos.\ x$.

6. Prove $\sin.\ (30^\circ + x) + \sin.\ (60^\circ + x) = \frac{1}{2}(1 + \sqrt{3})(\sin.\ x + \cos.\ x)$.

7. Prove $\cos.\ (30^\circ + x) + \cos.\ (60^\circ + x) = \frac{1}{2}(1 + \sqrt{3})(\cos.\ x - \sin.\ x)$.

8. Prove $\sin.\ (45^\circ + x) + \cos.\ (45^\circ + x) = \sqrt{2}\cos.\ x$.

9. Prove $\sin.\ (45^\circ + x) + \cos.\ (45^\circ + x) = \sin.\ (45^\circ - x) + \cos.\ (45^\circ - x)$.

10. Prove $\tan.\ (45^\circ + x) + \tan.\ (45^\circ - x) = \dfrac{2\,(1 + \tan.^2\ x)}{(1 - \tan.^2\ x)}$.

11. Prove $\tan.\ (x + 45^\circ) + \tan.\ (x - 45^\circ) = \dfrac{4\ \tan.\ x}{1 - \tan.^2\ x}$.

12. Show that $\sin.\ 75^\circ + \sin.\ 15^\circ = \sqrt{\frac{3}{2}}$.

13. Show that $\sin.\ 75^\circ - \sin.\ 15^\circ = \cos.\ 45^\circ$.

14. Show that $\cos.\ 22\frac{1}{2}^\circ - \cos.\ 67\frac{1}{2} = \sqrt{\dfrac{2 - \sqrt{2}}{2}}$.

15. Show that $\cos.\ 67\frac{1}{2}^\circ + \cos.\ 22\frac{1}{2} = \sqrt{\dfrac{2 + \sqrt{2}}{2}}$.

16. Show that $\sin.\ (90^\circ + x) = \cos.\ x$.

17. Show that $\cos.\ (90^\circ + x) = -\sin.\ x$.

18. Show that $\tan.\ (90^\circ + x) = -\cot.\ x$.

19. Derive the sine, cosine, and tangent of an angle of 18°.

*Suggestion.* Let $x = 18^\circ$; $5\,x = 90^\circ$; $2\,x = 36^\circ$; $3\,x = 54^\circ$; therefore, $\sin.\ 2\,x = \cos.\ 3\,x$. Expand both terms and solve the equation.

20. Derive the sine, cosine, and tangent of an angle of 15°.

*Suggestion.* Let $2\,x = 30^\circ$; then $x = 15^\circ$. Use Art. 14 and equations of Art. 75.

21. Derive the sine and cosine of an angle of 7° 30′.
22. Derive the sine, cosine, and tangent of an angle of 22° 30′.
23. Derive the sine and cosine of an angle of 168° 45′.
24. Derive the sine and cosine of an angle of 176° 15′.
25. Derive the sine and tangent of an angle of 9°.
26. Derive the sine and tangent of an angle of 24°.
27. Derive the cosine and secant of an angle of 6°.
28. Derive the sine and tangent of an angle of 168°.
29. Prove $\tan.^2 x \sec.^2 x - \sec.^2 x + 1 = \tan.^4 x$.
30. Prove $\sec.^2 x + \dfrac{\sin. x}{\cos.^2 x} = \dfrac{1}{1 - \sin. x}$.
31. Prove $\sec. x \operatorname{cosec}. x \tan. x = \sec.^2 x$.
32. Prove $\dfrac{a b \sec.^2 \theta}{a^2 - b^2 \tan.^2 \theta} = \dfrac{a b}{a^2 \cos.^2 \theta - b^2 \sin.^2 \theta}$.
33. Prove $\dfrac{n \sec.^2 \theta}{1 + n^2 \tan.^2 \theta} = \dfrac{n}{\cos.^2 \theta + n^2 \sin.^2 \theta}$.
34. If $\tan. 3x \cos.^2 3x = \tan. x \cos.^2 x$, find $x$. *Ans.* $x = 22° \, 30'$.
35. If $a - b \tan. x = (a - b) \sqrt{1 + \tan.^2 x}$, find $\tan. x$.

*Ans.* $+ \dfrac{-ab \pm (a - b) \sqrt{2ab}}{a^2 - 2ab}$.

36. If $\cos.^3 \phi = \sin. 2\phi \sin. \phi$, find $\phi$.
37. If $1 - a^2 \cos.^2 x = 2 a^2 \sin.^2 x$, find $\sin. x$ and $\cos. x$.
38. If $a^2 \cos.^2 x \sin.^2 x + \sin.^2 x + a^2 \cos.^4 x = \cos.^2 x$, find $\sin. x$.

*Ans.* $\text{Sin. } x = \pm \sqrt{\dfrac{1 - a^2}{2 - a^2}}$.

39. Given $\sin. x + \cos. 2x = \frac{3}{8}$; find $x$.
40. Given $\sec. x + \tan. x = \frac{5}{2}$; find $x$.
41. Given $\tan. x + \cot. x = 10$; find $x$.
42. Given $\sin. x + \cos. x = \frac{1}{3}$; find $x$.

# CHAPTER VIII.

## INVERSE TRIGONOMETRIC FUNCTIONS.—CIRCULAR MEASURE OF AN ANGLE.

Art. 76. If two quantities are so connected, that if one of them changes the other changes also, the second is said to be a *function* of the first.

Thus the sine, cosine, and tangent of any angle are functions of the angle, for when we change the angle the sine, cosine, tangent, etc., also change. In this sense, the *trigonometric ratios* of an angle are called its *trigonometric functions.*

If we consider the angle and any one of its functions as two *variables*, the angle will be the *independent* variable, and the function the *dependent* variable.

Thus in the equation $y = \sin. x$, when $x$ changes $y$ also changes, but the change of $y$ *depends* upon the change of $x$.

Now, if we wish to consider the angle as a function of one of its trigonometric ratios—that is, if we wish to represent the angle as the *dependent*, and the ratio as the *independent* variable, we use a system of *inverse notation*, in which the negative exponent $-1$ is employed.

In the equation $y = \sin. x$, $y$ is a function of $x$; but if we wish to express $x$ as a function of $y$, we indicate the relation between $y$ and $x$ in the form:

$$x = \sin.^{-1} y,$$

and read the equation, $x$ is an angle whose sine is $y$.

The relation between the angle and its other trigonometric ratios may be expressed in the same way.

Thus $x = \cos.^{-1} z$, $x = \tan.^{-1} w$, $z = \sec.^{-1} \frac{a}{b}$,

may be read; $x$ equals an angle whose cosine is $z$; $x$ equals an angle whose tangent is $w$; $z$ equals an angle whose secant is $\frac{a}{b}$.

The expressions $\sin.^{-1} y$, $\cos.^{-1} z$, $\tan.^{-1} w$, $\sec.^{-1} \frac{a}{b}$, etc., may stand alone, and are called *inverse trigonometric functions.*

They may be read: an angle whose sine is $y$, an angle whose cosine is $z$, etc.

77. *In a circle whose radius is unity, to find the length of an arc of any number of degrees, minutes, and seconds.*

Denote the circumference of a circle by $C$, its radius by $R$; then $C = 2\pi R$ (Ch. Art. 40, V.).

Making radius equal to unity, we have $C = 2\pi$.

That is, $2\pi$ is the length of the circumference of a circle whose radius is unity. As there are 360° in the whole circumference, $1° = \frac{1}{360}$ of $2\pi$, or $\frac{\pi}{180}$.

$$1' = \frac{1}{60} \text{ of } \frac{1}{360} \text{ of } 2\pi, \text{ or } \frac{\pi}{60 \times 180}.$$

$$1'' = \frac{1}{60} \text{ of } \frac{\pi}{60 \times 180}, \text{ or } \frac{\pi}{60 \times 60 \times 180}.$$

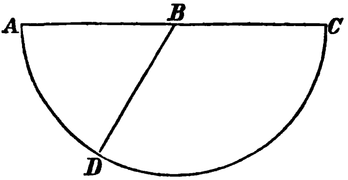
Denote the arc by $a$; then, generally, the length of an arc of $n$ degrees in a circle whose radius is unity ($n$ being any number, whole or fractional), is expressed by the equation,

$$a = \frac{n\pi}{180}.$$

78. In the ordinary measure of an angle and its intercepted arc by degrees, minutes, and seconds, the unit of an angle is the $\frac{1}{90}$th part of a right angle, and the unit of arc is the $\frac{1}{90}$th part of a quadrant (Ch. Art. 54, II.).

In the *circular measure* of an angle and its intercepted arc, *the unit of angle is an angle at the centre of a circle, subtended by an arc equal to the radius of the circle; and this arc is the unit of arc.*

79. *The angle at the centre of any circle subtended by an arc equal to the radius of the circle is a constant quantity.*



In the figure, suppose $AD$ to be an arc equal in length to the radius, $AB$, of the semicircle $ADC$. Denote the radius by $R$; $R$ will also represent the arc $AD$.

The length of the semicircle, $ADC$, equals $\pi R$ (Ch. Art. 40, V.).

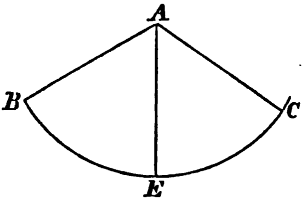
$$\frac{ABD}{2 \text{ right angles}} = \frac{AD}{ADC} \text{ (Euc. 33, VI. Ch. 19, II.)}$$

$$= \frac{R}{\pi R} = \frac{1}{\pi};$$

therefore $ABD = \frac{2 \text{ right angles}}{\pi}$.

But $\frac{2 \text{ right angles}}{\pi}$ is independent of the radius of the circle, and, therefore, a constant quantity.

Therefore the angle $ABD$, or an angle at the centre of any circle subtended by an arc equal to the radius of the circle, is a constant quantity.



80. Let $BAC$ be any angle at the centre of a circle, and be denoted by $A$. Denote the intercepted arc, $BC$, by $a$. Let $BE$, an arc equal to the radius of the circle, be denoted by $r$, and let $BAE$, the subtended unit of angle, be denoted by $u$.

Then $\frac{A}{u} = \frac{a}{r}$ (Euc. 33, VI. Ch. 19, II.); but, as $u$ is the unit of angle, we have

$$A = \frac{a}{r} \text{ (Ch. 20, II.).}$$

That is, the *circular measure of an angle is expressed by the ratio of the subtended arc to the radius of a circle described from the vertex of the angle as a centre.*

81. Four right angles at the centre of a circle are subtended by the whole circumference, or by $2\pi r$ (where $r$ denotes the radius). Therefore, by the preceding article,

$$4 \text{ right angles} = \frac{2\pi r}{r} = 2\pi; \text{ that is,}$$

the circular measure of 4 right angles is $2\pi$.

The circular measure of 2 right angles is $\frac{\pi r}{r} = \pi$; of 1 right angle is $\frac{\pi r}{2r} = \frac{\pi}{2}$; etc.

82. If, in the equation $A = \frac{a}{r}$, we let $r = 1$; that is, if we find the circular measure of an angle at the centre of a circle whose radius is unity, we derive the equation $A = a$; that is, *the circular measure of any angle, at the centre of a circle whose radius is unity, is equal to the length of the arc subtending that angle.*

83. Now, let $A$ be the circular measure of any angle, and $n$ the number of degrees, minutes, or seconds in the angle, $n$ being integral or fractional; then, by Arts. 82 and 77,

$$A = \frac{n\pi}{180};$$

which is the general expression for the circular measure of any angle, when the degrees, minutes, or seconds which it contains are given.

**In this system of measuring angles, an angle of:**

**4 right angles or an angle of 360°** $= \frac{360}{180}\pi$ **or** $2\pi$**; or**

**4 right angles** $= 2\pi = 6.2831854$**;**

**2 right angles** $= \frac{180}{180}\pi = \pi = 3.1415927$**;**

**1 right angle** $= \frac{90}{180}\pi = \frac{\pi}{2} = 1.5707963.$

an angle of 45° or $\frac{1}{2}$ right angle $= \frac{45}{180}\pi = \frac{\pi}{4} = .7853982$;

an angle of 60° $= \frac{60}{180}\pi = \frac{\pi}{3} = 1.0471976$;

an angle of 30° $= \frac{30}{180}\pi = \frac{\pi}{6} = .5235988$;

an angle of 1° $= \frac{\pi}{180} = .01745329$;

an angle of 1′ $= \frac{\pi}{60 \times 180} = .000290888$;

an angle of 1″ $= \frac{\pi}{60 \times 60 \times 180} = .00000485$.

84. *To find the number of degrees, minutes, and seconds in the unit angle of circular measure.*

In Art. 79 the unit angle is shown to be equal to $\frac{2 \text{ right angles}}{\pi}$.

2 right angles are measured by 180°:

Therefore the unit angle =

$$\frac{180°}{\pi} = \frac{180°}{3.1415927} = 57°.295779 = 57°17'44.8.''$$

85. *To change the measure of an angle in degrees, minutes, or seconds, to the equivalent circular measure; and conversely, to change the circular measure of an angle to its measure in degrees, minutes, or seconds.*

Let A be the circular measure of any angle, *n* the number of degrees, minutes, or seconds which it contains, *n* being any number, integral or fractional. According to Art. 83,

(1) $A = \frac{n\pi}{180}$; therefore,

(2) $n = A\frac{180}{\pi} = A \times 57.295779.$

To change degree measure into circular measure we use equation (1); to change circular measure into degree measure, we use equation (2).

1. Show that $\tan.^{-1} 1 + \tan.^{-1}(-1) = \pi$.

2. Show that $\sin.^{-1}\frac{1}{2} + \sin.^{-1}\frac{\sqrt{3}}{2} = \frac{\pi}{2}$.

3. If $\sin.^{-1} x + \sin.^{-1}\frac{2}{3}x = \frac{\pi}{3}$, find $x$. *Ans.* $x = \frac{3\sqrt{3}}{2\sqrt{19}}$.

4. Prove $\sin.^{-1}\frac{33}{65} = \sin.^{-1}\frac{4}{5} - \sin.^{-1}\frac{5}{13}$.

5. Prove $\sin.^{-1}\left(\frac{2\sqrt{15} + 7\sqrt{5}}{24}\right) = \sin.^{-1}\frac{2}{3} + \sin.^{-1}\frac{7}{8}$.

6. Prove $\sin.^{-1}\left(\frac{3\sqrt{5} + 2\sqrt{7}}{12}\right) = \sin.^{-1}\frac{3}{4} + \sin.^{-1}\frac{2}{3}$.

7. Prove $\tan.^{-1}\frac{29}{14} = \tan.^{-1}\frac{6}{5} + \tan.^{-1}\frac{1}{4}$.

8. Prove $\tan.^{-1}\frac{16}{17} = \tan.^{-1}\frac{13}{7} - \tan.^{-1}\frac{1}{3}$.

9. Find the number of degrees, minutes, and seconds, in an angle whose circular measure is $\frac{\pi}{14}$. *Ans.* 12° 51′ 25.7″.

10. Find the number of degrees, minutes, and seconds, in an angle whose circular measure is $\frac{2\pi}{13}$.

11. Find the number of degrees, minutes, and seconds, in an angle whose circular measure is $\frac{6}{5}$, that is, $\frac{6}{5}$ of the unit angle.

*Ans.* 68° 45′ 17.77″.

12. Find the number of degrees, minutes, and seconds, in an angle of which the circular measure is .763.

13. What is the circular measure of an angle of 11° 15′ ? *Ans.* $\frac{\pi}{16}$.

14. What is the circular measure of an angle of 1° 30′ ?

15. What is the circular measure of an angle of 7° 30′ ?

16. What is the number of degrees, minutes, and seconds, in an arc 40 feet long, in a circle whose radius is 35 feet?

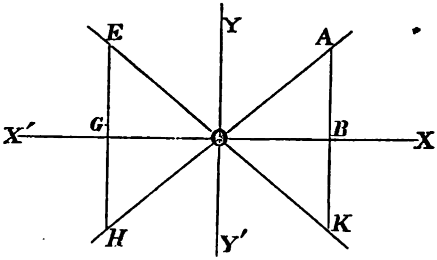
*Solution.* Degrees in arc = degrees in angle = unit × $\frac{\text{arc}}{\text{radius}} = \frac{180°}{\pi} \times \frac{40}{35} = 57°.295779 \times \frac{8}{7} = 65°\ 28'\ 51''.2.$

17. What is the number of degrees, minutes, and seconds in an angle subtended by an arc of 25 feet in a circle whose radius is 15 feet?

# CHAPTER IX.

ANGLES HAVING CORRESPONDING TRIGONOMETRIC FUNCTIONS OF THE SAME NUMERICAL VALUE.—TABLE OF IMPORTANT TRIGONOMETRIC FUNCTIONS.—ANGLES GREATER THAN FOUR RIGHT ANGLES.—POSITIVE AND NEGATIVE ANGLES.

Art. 86. *Angles having corresponding trigonometric functions of the same numerical value.*



In the figure, let $XX'$ and $YY'$, two lines at right angles at O, be the *initial* (Art. 41) lines.

Beginning with $XOY$, and going from right to left, call $XOY$ the *first quadrant*, $YOX'$ the *second quadrant*, $X'OY'$ the *third quadrant*, and $Y'OX$ the *fourth quadrant.*

Let two lines, $AH$ and $EK$, be drawn making equal angles with $XX'$ and intersecting $XX'$ and one another at the same point, O. Then will the angles $XOA$, $EOX'$, $X'OH$, and $KOX$ be equal to one another.

If, now, we take the points $A$, $E$, $H$, and $K$, equally distant from $O$, and join the points $A$ and $K$ by the straight line $AK$, and the points $E$ and $H$ by the straight line $EH$, we shall have four equal right-angled triangles, $BOA$, $EOG$, $GOH$, and $KOB$ (Euc. 4, I. Ch. 20, I).

Besides these four right-angled triangles, there can be no others equal to them, having their bases and perpendiculars in the same relative position, with respect to the initial lines and the point $O$.

Reckoning the angles from $OX$ (from right to left) through the four quadrants, there will be, therefore, only *four* angles, having equal *triangles of reference* (Art. 3).

The triangle of reference for an *acute* angle will be in the *first quadrant;* for an *obtuse* angle, in the *second quadrant;* for an angle *greater than two right angles* but *less than three right angles*, in the *third quadrant;* for an *angle greater than three right angles but less than four right angles*, in the *fourth quadrant.*

The *numerical values of the six trigonometric functions*, sine, cosine, tangent, cotangent, secant, and cosecant, *of these four angles will be the same* (Euc. 4, VI. Ch. 4, III).

Thus the angles $BOA$, $BOE$, and the salient angles $BOH$ and $BOK$, will have equal triangles of reference, and, consequently, equal numerical values for the corresponding trigonometric functions, sine, cosine, tangent, cotangent, secant, and cosecant.

87. *To find expressions for the four angles whose trigonometric functions have the same numerical value.*

In the figure, denote the angle $BOA$ by $a$, then each of the equal angles, $EOG$, $GOH$, and $KOB$, will also be denoted by $a$.

$BOA$, $BOE$, the salient angles $BOH$ and $BOK$, are the angles which have the same numerical values for their corresponding trigonometric functions.

$BOA = a.$

$BOE = 2$ right angles $- EOG = 180° - a = \pi - a.$

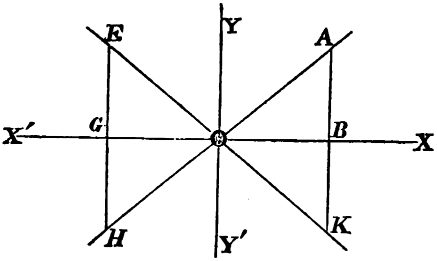
$BOH = 2$ right angles $+ GOH = 180° + a = \pi + a.$

$BOK = 4$ right angles $- KOB = 360° - a$
$= 2\pi - a.$

Denoting then any acute angle by $a$, the four angles, which have the same numerical values for their corresponding trigonometric functions, are:

$$a,\ 180° - a,\ 180° - a,\ 360° - a;\ \text{or,}$$
$$a,\ \pi - a,\ \pi + a,\ 2\pi - a.$$

88. *To find the signs of the trigonometric functions of angles having equal triangles of reference.*



In the figure, let the construction be the same as in Art. 86.

In the first quadrant, the *perpendicular* and *base* of the triangle of reference, $AOB$, are *positive* (Art. 41);

therefore, as the hypotenuse is positive, *all the trigonometric functions of an angle in the first quadrant are positive*, all the ratios having like signs in both their terms.

In the second quadrant, $YOX'$, the *perpendicular* is always *positive*, while the *base* is always *negative;* therefore, where the triangle of reference is in the second quadrant, the trigonometric functions of the angle in which the *perpendicular* occurs with the *hypotenuse* are *positive*, both terms of the ratios having like signs; and the trigonometric functions in which the *base* occurs are *negative*, as the two terms of these ratios have opposite signs.

That is, of an obtuse angle the *sine* and *cosecant* are *positive*, but the *cosine*, *tangent*, *cotangent*, and *secant are negative.*

In the third quadrant, $X'OY'$, the *perpendicular* and *base* are both *negative;* therefore, where the triangle of reference is in the *third* quadrant, the trigonometric functions of the angle in which the *perpendicular* or *base* occur with the *hypotenuse* are *negative;* but those in which the *base* and *perpendicular* occur together are *positive.*

That is, of an angle greater than two right angles and less than three right angles, or greater than $\pi$ and less than $\frac{3\pi}{2}$, the *sine*, *cosine*, *secant*, and *cosecant* are *negative*, but the *tangent* and *cotangent* are *positive.*

In the fourth quadrant, $Y'OX$, the *perpendicular* is always *negative*, while the *base* is *positive;* therefore, when the triangle of reference is in the *fourth* quadrant, the trigonometric functions of the angle in which the *perpendicular* occurs are *negative;* and the trigonometric functions in which the *base* occurs with the *hypotenuse* are *positive.*

That is, of an angle greater than three right angles and less than four right angles, or greater than $\frac{3\pi}{2}$ and less than $2\pi$, the *sine, tangent, cotangent,* and *cosecant* are *negative,* but the *cosine* and *secant* are *positive.*

The versed sine is always positive.

The following table will show the results of the preceding article:

| ANGLE. | SIN. | TAN. | SEC. | COSIN. | COTAN. | COSEC. | VERSIN. |
|---|---|---|---|---|---|---|---|
| $> 0$ and $< \frac{\pi}{2}$ | + | + | + | + | + | + | + |
| $> \frac{\pi}{2}$ and $< \pi$ | + | − | − | − | − | + | + |
| $> \pi$ and $< \frac{3\pi}{2}$ | − | + | − | − | + | − | + |
| $> \frac{3\pi}{2}$ and $< 2\pi$ | − | − | + | + | − | − | + |

89. *Particular angles having equal triangles of reference.*

(1) In the figure of Art. 86, let $A\,O\,B$ or $a$, equal 0; then the expressions, $a$, $\pi - a$, $\pi + a$, $2\pi - a$ (Art. 87), reduce to 0, $\pi$, and $2\pi$;

Therefore the corresponding trigonometric functions of 0, $\pi$, and $2\pi$, are equal in *numerical* value (Art. 39).

As the perpendicular of the triangle of reference disappears at 0, at $\pi$, and at $2\pi$, *the signs* of the trigonometric ratios, in which the *perpendicular* occurs, will be *undetermined;* but, as the base remains, the signs of the ratios in which the *base* occurs with the hypotenuse will be *determined* by the sign of the base, and will therefore be *positive* at 0 and $2\pi$, but *negative* at $\pi$.

Let $A\ O\ B$ or $a$ equal $\frac{\pi}{2}$; then the expressions

$a$, (2) $\pi - a$, $\pi + a$, $2\pi - a$, reduce to $\frac{\pi}{2}$ and $\frac{3\pi}{2}$; therefore the corresponding trigonometric functions of $\frac{\pi}{2}$ and $\frac{3\pi}{2}$ are the same in *numerical* value (Arts. 35, 36, 37, 38).

As the base of the triangle of reference disappears at $\frac{\pi}{2}$ and at $\frac{3\pi}{2}$, the *signs* of the ratios in which the *base* occurs will be *undetermined;* but as the perpendicular remains, the signs of the ratios in which the *perpendiculr* occurs with the *hypotenuse*, will be determined by the sign of the perpendicular, and will therefore be *positive* at $\frac{\pi}{2}$, but *negative* at $\frac{3\pi}{2}$.

Let $a = \frac{\pi}{6}$; then, by Art. 87,

(3) $\frac{\pi}{6}$, $\frac{5\pi}{6}$, $\frac{7\pi}{6}$, $\frac{11\pi}{6}$, will have their corresponding trigonometric functions the same in numerical value (Art. 14).

Let $a = \frac{\pi}{4}$; then,

(4) $\frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}, \frac{7\pi}{4}$, will have their corresponding trigonometric functions the same in numerical value (Art. 15).

Let $a = \frac{\pi}{3}$; then,

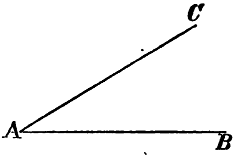
(5) $\frac{\pi}{3}, \frac{2\pi}{3}, \frac{4\pi}{3}, \frac{5\pi}{3}$, will have their corresponding trigonometric functions the same in numerical value (Art. 16).

The signs of the functions (3), (4), and (5) will be determined by Art. 88.

(90) The following table will show the results of the preceding article:

| Angle....... | 0° | 30° | 45° | 60° | 90° | 120° | 135° | 150° | 180° | 210° | 225° | 240° | 270° | 300° | 315° | 330° | 360° |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | $0$ | $\frac{\pi}{6}$ | $\frac{\pi}{4}$ | $\frac{\pi}{3}$ | $\frac{\pi}{2}$ | $\frac{2\pi}{3}$ | $\frac{3\pi}{4}$ | $\frac{5\pi}{6}$ | $\pi$ | $\frac{7\pi}{6}$ | $\frac{5\pi}{4}$ | $\frac{4\pi}{3}$ | $\frac{3\pi}{2}$ | $\frac{5\pi}{3}$ | $\frac{7\pi}{4}$ | $\frac{11\pi}{6}$ | $2\pi$ |
| Sine.......... | $0$ | $\frac{1}{2}$ | $\frac{\sqrt{2}}{2}$ | $\frac{\sqrt{3}}{2}$ | $1$ | $\frac{\sqrt{3}}{2}$ | $\frac{\sqrt{2}}{2}$ | $\frac{1}{2}$ | $0$ | $-\frac{1}{2}$ | $-\frac{\sqrt{2}}{2}$ | $-\frac{\sqrt{3}}{2}$ | $-1$ | $-\frac{\sqrt{3}}{2}$ | $-\frac{\sqrt{2}}{2}$ | $-\frac{1}{2}$ | $0$ |
| Tangent..... | $0$ | $\frac{\sqrt{3}}{3}$ | $1$ | $\sqrt{3}$ | $\infty$ | $-\sqrt{3}$ | $-1$ | $-\frac{\sqrt{3}}{3}$ | $0$ | $\frac{\sqrt{3}}{3}$ | $1$ | $\sqrt{3}$ | $\infty$ | $-\sqrt{3}$ | $-1$ | $-\frac{\sqrt{3}}{3}$ | $0$ |
| Secant....... | $1$ | $\frac{2\sqrt{3}}{3}$ | $\sqrt{2}$ | $2$ | $\infty$ | $-2$ | $-\sqrt{2}$ | $-\frac{2\sqrt{3}}{3}$ | $-1$ | $-\frac{2\sqrt{3}}{3}$ | $-\sqrt{2}$ | $-2$ | $\infty$ | $2$ | $\sqrt{2}$ | $\frac{2\sqrt{3}}{3}$ | $1$ |
| Cosine....... | $1$ | $\frac{\sqrt{3}}{2}$ | $\frac{\sqrt{2}}{2}$ | $\frac{1}{2}$ | $0$ | $-\frac{1}{2}$ | $-\frac{\sqrt{2}}{2}$ | $-\frac{\sqrt{3}}{2}$ | $-1$ | $-\frac{\sqrt{3}}{2}$ | $-\frac{\sqrt{2}}{2}$ | $-\frac{1}{2}$ | $0$ | $\frac{1}{2}$ | $\frac{\sqrt{2}}{2}$ | $\frac{\sqrt{3}}{2}$ | $1$ |
| Cotangent... | $\infty$ | $\sqrt{3}$ | $1$ | $\frac{\sqrt{3}}{3}$ | $0$ | $-\frac{\sqrt{3}}{3}$ | $-1$ | $-\sqrt{3}$ | $\infty$ | $\sqrt{3}$ | $1$ | $\frac{\sqrt{3}}{3}$ | $0$ | $-\frac{\sqrt{3}}{3}$ | $-1$ | $-\sqrt{3}$ | $\infty$ |
| Cosecant..... | $\infty$ | $2$ | $\sqrt{2}$ | $\frac{2\sqrt{3}}{3}$ | $1$ | $\frac{2\sqrt{3}}{3}$ | $\sqrt{2}$ | $2$ | $\infty$ | $-2$ | $-\sqrt{2}$ | $-\frac{2\sqrt{3}}{3}$ | $-1$ | $-\frac{2\sqrt{3}}{3}$ | $-\sqrt{2}$ | $-2$ | $\infty$ |

(91). *Angles greater than four right angles.*



If the line $A\ C$ be conceived as first coinciding with $A\ B$, and then to revolve about $A$, from right to left, till it again coincide with $A\ B$, it will pass through four right angles. If, after coinciding with $A\ B$, it still continues to revolve, from right to left, till it takes the position of $A\ C$ (in the figure), it is said to make with $A\ B$ *an angle greater than four right angles.*

Let $C A B$ be denoted by $a$. Then the angle, greater than four right angles, which $C A$ makes with $A\ B$, will be expressed by $2\pi + a$.

If $C A$ be revolved twice through four right angles, and then continue to revolve till it takes the position $A\ C$; the angle, greater than eight right angles, which $A\ C$ makes with $A\ B$, will be expressed by $4\pi + a$.

Generally an angle greater than four right angles may be expressed by $2n\pi + a$, where $n$ is any integral, and $a$ is the excess of the angle over $2n\pi$.

92. *Trigonometric functions of angles greater than four right angles.*

It is evident, from the figure, that the trigonometric functions of the angle $2n\pi + a$ will be the same as those of $a$, as they will have the same triangle of reference.

*Therefore, to find the trigonometric ratios of an angle greater than four right angles, we subtract from the angle four right angles, or some multiple of four right angles, and take the trigonometric ratios of the remaining angle.*

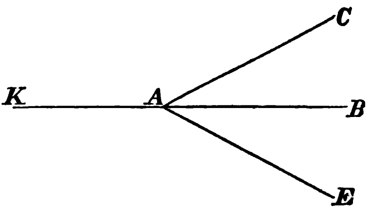
Thus $\sin.\ (2n\pi + \alpha) = \sin.\ \alpha.$
$\tan.\ (2n\pi + \alpha) = \tan.\ \alpha.$
$\sec.\ (2n\pi + \alpha) = \sec.\ \alpha.$
$\cos.\ (2n\pi + \alpha) = \cos.\ \alpha.$
$\cot.\ (2n\pi + \alpha) = \cot.\ \alpha.$
$\text{cosec.}\ (2n\pi + \alpha) = \text{cosec.}\ \alpha.$
$\text{versin.}\ (2n\pi + \alpha) = \text{versin.}\ \alpha.$

93. It has been shown that an acute angle, an obtuse angle, an angle greater than two right angles, an angle greater than three right angles, an angle greater than four right angles, all have equal triangles of reference (Arts. 86, 91).

Therefore the relations (established for acute and obtuse angles, in Chapter VII.), of trigonometrical functions of the same angle to each other, and the relations of the trigonometrical functions of the sum or difference of two angles to the functions of the angles, and the relations following these, may also be considered as established for any angle whatever, as all these relations depend upon the triangles of reference.

94. *Positive and negative angles.*

If a horizontal line, $K B$, be taken as an initial line, and a straight line, as $A C$, be made to coincide with $A B$, and then to revolve about $A$ *from right to left*, assuming various positions, the angles which it then makes with $A B$ are *positive* angles.

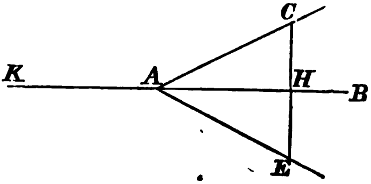


If the line $A C$ be revolved about $A$ in a contrary direction (i. e., as the hand of a watch), the angles which it then makes with $A B$ are *negative angles.*

Thus, in the figure, $BAC$ is a *positive* angle; $BAE$ is a *negative* angle.

95. *Relations between the trigonometric functions of positive angles, and of equal negative angles.*

In the figure, let $KB$ be the initial line; $BAC$ be a positive angle; and $BAE$ an equal negative angle. Denote the angle $BAC$ by $a$; then $BAE$ will be denoted by $-a$.



From $C$, any point in $AC$, draw a perpendicular to $AB$, and produce it to meet $AE$ in $E$.

The two triangles of reference, $CAH$ and $HAE$, are equal in all respects, except that the sign of $EH$ is negative, while the sign of $CH$ is positive; therefore the trigonometric functions of $-a$ will be numerically equal to those of $a$, but the *signs* of the functions of the two angles in which the perpendicular occurs will be unlike, while the signs of the functions in which the base occurs with the hypotenuse will be the same (Art. 41).

Thus, $$\text{sin.}\,(-a)=\frac{EH}{AE}=\frac{-CH}{CA}=-\text{sin.}\,a;$$

$$\text{tan.}\,(-a)=\frac{EH}{AH}=\frac{-CH}{AH}=-\text{tan.}\,a;$$

$$\text{sec.}\,(-a)=\frac{AE}{AH}=\frac{AC}{AH}=\text{sec.}\,a;$$

$$\text{cos.}\,(-a)=\frac{AH}{AE}=\frac{AH}{AC}=\text{cos.}\,a;$$

$$\text{cot.}\,(-a)=\frac{AH}{EH}=\frac{AH}{-CH}=-\text{cot.}\,a;$$

$$\text{Cosec.}\,(-a)=\frac{AE}{EH}=\frac{AC}{-CH}=-\,\text{cosec.}\,a;$$

$$\text{Versin.}\,(-a)=1-\frac{AH}{AE}=1-\frac{AH}{AC}=\text{versin.}\,a.$$

**Example** 1. Find the sine and cosine of $\frac{9\pi}{8}$, and of $\frac{15\pi}{8}$.

2. Find the sine and cosine of $a$, $\pi-a$, $\pi+a$, and $2\pi-a$, where $a=15°$.

3. What angles are expressed in degrees by $\pi(n \pm \frac{1}{6})$, $n$ being 0 or any integral number? Write the sines of these angles.

4. Write the cosines for the series of angles expressed by $\pi(n \pm \frac{1}{3})$, $n$ being 0, or any integral.

5. Write the tangents of the series of angles $\pi(n \pm \frac{1}{4})$, $n$ being 0, or any integral.

6. Write the trigonometric ratios for the angles $\frac{(2n+1)\pi}{}$, $n$ being 0, or any integral.

7. Prove $\dfrac{\sec.^2\left(\frac{\pi}{4}+\frac{x}{2}\right)}{2\tan.\left(\frac{\pi}{4}+\frac{x}{2}\right)}=\dfrac{1}{\cos.\,x}$.

8. Prove $1+\dfrac{\sin.\left(\frac{\pi}{4}-x\right)}{\cos.\left(\frac{\pi}{4}-x\right)}=\dfrac{2}{1+\tan.\,x}$.

9. Prove that $x+\frac{1}{2}y=\pi$, if $\dfrac{\sin.\,y}{1-\cos.\,y}+\dfrac{\cos.\,x}{\sin.\,x}=0$.

10. Find two circular measures of $\theta$ which will satisfy the equation $\cos.\,\theta+\cos.^2\theta=\sin.^2\theta$. *Ans.* $\theta=\frac{\pi}{3}$ and $\theta=\pi$.

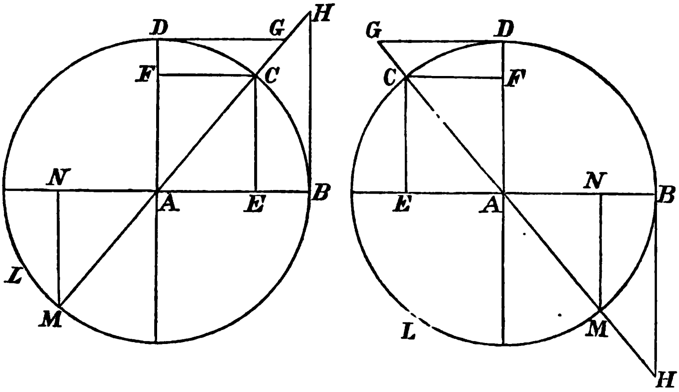
11. Find two circular measures of $\theta$ which will satisfy the equation $2\cos.\,\theta\sin.\,2\theta-2\cos.\,2\theta=2\cos.\,2\theta\sin.\,\theta$. *Ans.* $\theta=\frac{3\pi}{2}$ and $\frac{\pi}{6}$.

12. Find one circular measure of $\phi$ which will satisfy the equation $3\sin.^2\phi+2\sin.\,\phi=1$.

# CHAPTER X.

## TRIGONOMETRIC FUNCTIONS REPRESENTED BY LINES.— OUTLINE OF THE METHOD OF CONSTRUCTING TRIGONOMETRIC TABLES.

Art. 96. The trigonometric functions of an angle are sometimes represented by lines having certain positions in, or with respect to, a circle described from the vertex of the angle as a centre, and with a radius regarded as a unit of length.



In the figure, let $B A C$ be any angle.

From the vertex $A$, as a centre, with any unit of length as a radius, describe the circle $B C L$, intersecting the sides of the angle at $B$ and $C$. At $A$ draw $D A$ at right angles to $A B$. From $C$ draw $C E$ at right angles to

$AB$ or $AB$ produced. From $B$ draw the tangent $BH$ meeting $AC$ produced at $H$. Also, from $C$ draw $CF$ perpendicular to $AD$ and from $D$ draw the tangent $DG$.

Let $AC$, produced through the centre, meet the circumference at $M$, and the tangent $BH$ at $H$.

From $M$ draw $MN$ perpendicular to $AB$, or $AB$ produced.

The *sine* of an angle (or its subtending arc), is the perpendicular drawn from one extremity of the arc to the radius, or radius produced, passing through the other extremity of the arc.

Thus, sin. $BAC$ or sin. $BC = CE$.

The trigonometric *tangent* of an angle (or its subtending arc), is the part of the geometric tangent at one extremity of the arc, terminated by the point of contact and the produced radius passing through the other extremity of the arc.

Thus, tan. $BAC$ or tan. $BC = BH$.

The *secant* of an angle (or its subtending arc), is the part of the produced radius between the centre and the tangent.

Thus, sec. $BAC$ or sec. $BC = AH$.

The *cosine*, *cotangent*, and *cosecant* of an angle, (or its subtending arc), are the sine, tangent, and secant, respectively, of the complement of the angle or arc.

Thus: cos. $BAC$, or cos. $BC =$ sin. $CAD$
$=$ sin. $CD = CF$;

cotan. $BAC$ or cot. $BC =$ tan. $CAD =$ tan. $CD$
$= DG$;

cosec. $BAC$ or cosec. $BC =$ sec. $CAD =$ sec. $CD$
$= AG$.

The versed sine of an angle (or arc) is the distance between the foot of the sine (drawn from one extemity of the arc) and the other extremity of the arc, measured on the radius, or radius produced.

Thus, versin. $BAC$, or versin. $BC = BE$.

It is evident from the figure that the sines and cosines of all angles not 0, or not multiples of a right angle, are less than the radius, that is, less than 1 (1 standing for the unit of length) (Euc. 15, III. Ch. 3, II.).

By bisecting the arc of the quadrant $BD$, and drawing tangents, it can be shown that a tangent may be less than radius, equal to radius, or greater than radius; that is, less than 1, equal to 1, or greater than 1 (Euc. 5 and 6, I. Ch. 27 and 28, I.).

It is also evident from the figure, that the secants of all angles not 0, or not equal to two or to four right angles, are greater than radius, that is, greater than 1.

At 0° the sine is 0, the tangent is 0, the secant 1, and the cosine 1.

At 90° the sine is 1, the tangent $\infty$, the secant $\infty$, and the cosine 0.

97. Lines from *above*, perpendicular upon $AB$, or $AB$ produced are *positive;* and lines from *below*, perpendicular to $AB$ are *negative.*

Perpendicular lines proceeding from $AD$, or $AD$ produced, to the *right* are *positive*, while those proceeding to the *left* are *negative.*

Also, the radius and lines measured on the radius or radius produced, *measured from* the centre *to* or *through* one *extremity* of the arc are positive; but lines measured on the radius produced through the centre in the *opposite direction* from the *extremity* of the *arc* are *negative.*

Thus, all the lines representing the trigonometric functions of an acute angle are positive.

Of the lines representing the trigonometric functions of an obtuse angle, the sine and cosecant are positive; but the cosine, tangent, cotangent, and secant are negative.

That is, (in the right-hand figure, page 105), $CE$ and $AG$ are positive; $AE$, $BH$, $DG$, and $AH$ are negative.

If (in the left-hand figure) we have an angle, $BAM$ (salient), greater than two right angles:

its sine $MN$, cosine $AN$, secant $AH$, and cosecant $AG$ are negative; but the tangent $BH$ and cotangent $DG$ are positive.

If (in the right-hand figure) we have a salient angle, $BAM$, greater than three right angles:

its sine $MN$, tangent $BH$, cotangent $DG$, and cosecant $AG$ are negative; but the cosine $AN$ and secant $AH$ are positive.

The versed sine in all cases is positive.

The results of this article will, on comparison, be seen to agree with the results of Art. 88.

98. From the similar triangles $BAH$ and $EAC$ (*see* figure on page 105),

$$BH : CE :: BA : AE \text{ (or } CF\text{)};$$

therefore, $BH = \frac{BA \times CE}{CF}$; or, denoting the angle $BAC$ by $A$, and the radius by $R$, $\tan. A = \frac{R \sin. A}{\cos. A}$;

but, as $R = 1$, $\tan. A = \dfrac{\sin. A}{\cos. A}$.

In a similar manner $AH : AC :: AB : AE$;

therefore $AH = \sec. A = \dfrac{R^2}{\cos. A} = \dfrac{1}{\cos. A}$.

From similar triangles, $ACF$ and $AGD$,

$$AG : AC :: AD : AF (= CE);$$

therefore, $AG = \text{cosec}. A = \dfrac{R^2}{\sin. A} = \dfrac{1}{\sin. A}$.

In a similar manner:—

$$DG : FC :: AD : AF (= CE);$$

therefore, $DG = \cot. A = \dfrac{R \cos. A}{\sin. A} = \dfrac{\cos. A}{\sin. A}$.

Again, from similar triangles, $ADG$ and $ABH$,

$$DG : AB :: DA : BH;$$

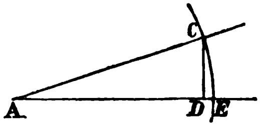
therefore, $DG = \cot. A = \dfrac{R^2}{\tan. A} = \dfrac{1}{\tan. A}$.

These results have already been obtained by the method of ratios (Art. 64).

99. In Arts. 14 and 15 we have obtained the trigonometric functions of an angle of 30° and of 45° respectively. By means of the solution of Example 19 at the end of Chapter VII., we can find the trigonometric functions of an angle of 18°. By Art. 75 we can then obtain, from these functions, the functions of an angle of 15°, 22½°, and 9°; and then the functions of an angle of 7° 30′, 11° 15′, 4° 30′; and then the functions of the halves of these angles, and so on. Also by means of Arts. 67, 68, and 72, we can obtain the functions of the sums and differences of the angles thus found.

By these methods, however, we can ascertain the trigonometric functions of only a part of the angles of the first quadrant.

By the method given in outline in the following article, we can ascertain the trigonometric functions of any angle of the quadrant.



100. *Outline of a method of constructing tables of trigonometric functions of angles of the first quadrant.*

Let $A$ be a small acute angle; $CAD$ its triangle of reference.

From $A$ as a centre with a radius $AC$ describe the arc $CE$, intersecting the sides of the angle in $C$ and $E$.

If $BAC$ were one of a number of equal angles at the centre of a circle, $CD$ would be half the side of a regular polygon inscribed in the circle. By increasing the number of sides of the polygon, that is, by decreasing the size of the angle at the centre, $CD$ would very nearly equal the arc $CE$ (Ch. 14, V.). Consequently, if the angle A were small enough, $\frac{CD}{CA}$ would approximately equal $\frac{CE}{CA}$. But $\frac{CD}{CA} = \sin. A$ (1), (Art. 4), and $\frac{CE}{CA} =$ the circular measure of $A$ (Art. 80); therefore, of *a very small angle the sine approximately equals the circular measure.*

In the construction of trigonometric tables, the angle $A$ is taken as very small, and the sine of the angle is taken as *equal* to its circular measure. The angle is sometimes taken as $1'$ and sometimes as $10''$.

Suppose, then, $A$ equal $1'$, and that its sine is taken as equal to its circular measure; then:

Sin. $1' = .00029089$;

Cos. $1' = \sqrt{1 - \sin.^2 1'} = \sqrt{(1 + \sin. 1')(1 - \sin. 1')}$
$= \sqrt{1.00029089 \times .99970911} = .999999957.$

Now by equations ($a$) and ($c$) of Art. 69,

Sin. $(x + y) = 2 \sin. x \cos. y - \sin. (x - y)$.

Cos. $(x + y) = 2 \cos. x \cos. y - \cos. (x - y)$.

In these equations let $y = 1'$, and let $x$ be in succession $1'$, $2'$, $3'$, etc., and we shall have for the sines

(1) Sin.$2' = 2 \sin. 1' \cos. 1'$;
(2) Sin.$3' = 2 \sin. 2' \cos. 1' - \sin. 1'$;
(3) Sin.$4' = 2 \sin. 3' \cos. 1' - \sin. 2'$, etc.;

and for the cosines

(4) Cos.$2' = 2 \cos.^2 1' - 1$;
(5) Cos.$3' = 2 \cos. 2' \cos. 1' - \cos. 1'$;
(6) Cos.$4' = 2 \cos. 3' \cos. 1' - \cos. 2'$; etc.

By substituting in equation (1) for sin. $1'$, its value, and for cos. $1'$ its value, we obtain sin. $2'$; then, by substitution of values in (2) we obtain sin. $3'$; and so on.

By substituting in equation (4) for cos. $1'$ its value, we obtain cos. $2'$; and then by substitution of values in (5) we obtain cos. $3'$; and so on.

Thus, starting with the sine and the cosine of $1'$, we find the sine and cosine of $2'$, of $3'$, of $4'$, etc.

The tangents, cotangents, secants, and cosecants are derived from the sines and cosines by means of the equations of Art. 64.

Thus, tan. $2' = \dfrac{\sin. 2'}{\cos. 2'}$; sec. $2' = \dfrac{1}{\cos. 2'}$; etc.

It is not necessary to obtain the functions of all the angles of the quadrant by the above method. For instance, suppose we have obtained the functions of angles up to 45°, at intervals of 1′. Then, denoting by $x$ the excess of any angle over 45°, we have—

$$\sin.\ (45^\circ + x) = \cos.\ (45^\circ - x);$$
$$\cos.\ (45^\circ + x) = \sin.\ (45^\circ - x);$$
$$\tan.\ (45^\circ + x) = \cot.\ (45^\circ - x)\ (\text{Art. } 5);$$

since $45^\circ + x$ is the complement of $45^\circ - x$.

Let the angle, whose functions are required, be 47° 35′. Then

$$x = 2^\circ\ 35', \text{ and}$$
$$\sin.\ 47^\circ\ 35' = \cos.\ 42^\circ\ 25';$$
$$\cos.\ 47^\circ\ 35' = \sin.\ 42^\circ\ 25';$$
$$\tan.\ 47^\circ\ 35' = \cot.\ 42^\circ\ 25'; \text{ etc.}$$

The right-hand members of these equations are supposed to be already obtained. Therefore we shall know their equals, sin. 47° 35′, etc.

The results obtained by the methods of Art. 99 may also be used to test the accuracy of the work by the method of this article.

THE END.

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